Effects of Analytical and Numerical Jacobians on Aerodynamic Design

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Outline

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Introduction

- The most important part of aerodynamic design optimization is accurate and efficient computation of the sensitivity derivatives.
- *Analytical sensitivity derivatives* are more accurate than ones obtained by finite-difference methods.
- Efficiency of analytical sensitivity derivative calculation can be improved by using *direct flow solvers.*
Direct solvers require the calculation of *Jacobian matrix*.

Derivation of *analytical Jacobians* becomes more difficult as the discretization of governing equations become more complex.

The best alternative is to compute the Jacobians *numerically* as accurate as possible.
Objectives

- To compare the accuracy of numerical and analytical Jacobians used in direct flow solvers
- To investigate their effects on aerodynamic design optimization
- To improve the efficiency of Jacobian matrix solution with parallel processing
Flow Analysis

- Flow code under development is a **2-D Euler solver**
  - Newton's method
  - upwind flux splitting schemes
  - both numerical and analytical Jacobian matrices
- Different *geometries, grid sizes and BCs* will be considered.
- *Parallel processing* will be used in solving the Jacobian matrix.
- Code will be improved to be a **2-D / Axisymmetric Navier-Stokes solver**.
Flow Analysis (Cont.)

- **2-D steady Euler equations** in generalized coordinates:

\[
\frac{\partial \hat{F}(\hat{Q})}{\partial \xi} + \frac{\partial \hat{G}(\hat{Q})}{\partial \eta} = 0
\]

where

\[
\hat{Q} = J^{-1} \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e_t \end{bmatrix}, \quad \hat{F} = J^{-1} \begin{bmatrix} \rho U \\ \rho u U + \xi_x p \\ \rho v U + \xi_y p \\ (\rho e_t + p) U \end{bmatrix}, \quad \hat{G} = J^{-1} \begin{bmatrix} \rho V \\ \rho u V + \eta_x p \\ \rho v V + \eta_y p \\ (\rho e_t + p) V \end{bmatrix}
\]
Flow Analysis (Cont.)

- **Newton’s method:**

  \[ R(Q) = 0 \]
  
  \[
  \left( \frac{\partial R}{\partial Q} \right)^n \cdot \Delta Q^n = -R(Q^n) \]

  \[ Q^{n+1} = \Delta Q^n + Q^n \]

- **Numerical Jacobians:**

  \[
  \frac{\partial R_i}{\partial Q_j} = \frac{R_i(Q + \varepsilon \cdot e_j) - R_i(Q)}{\varepsilon}
  \]
Sensitivity Analysis

- **Analytical sensitivity derivatives** will be obtained by using both numerical and analytical Jacobian matrices for converged solutions.
- Obtained sensitivity derivatives will be used in *Euler and Navier-Stokes* aerodynamic design optimization applications.
Sensitivity Analysis (Cont.)

- **Standart formulation** for optimization:

\[
F(Q(x_D), X(x_D), x_D) = 0 \quad \text{(Objective Function)}
\]

\[
G(Q(x_D), X(x_D), x_D) = 0 \quad \text{(Constraint Function)}
\]
Sensitivity Analysis (Cont.)

- **Analytical sensitivity derivative** calculation:

\[
\frac{dF}{dx_D} = \frac{\partial F}{\partial Q} \cdot \left( \frac{dQ}{dx_D} \right) + \frac{\partial F}{\partial X} \cdot \left( \frac{dX}{dx_D} \right) + \frac{\partial F}{\partial x_D}
\]

\[
\frac{dG}{dx_D} = \frac{\partial G}{\partial Q} \cdot \left( \frac{dQ}{dx_D} \right) + \frac{\partial G}{\partial X} \cdot \left( \frac{dX}{dx_D} \right) + \frac{\partial G}{\partial x_D}
\]
Sensitivity Analysis (Cont.)

- **Analytical sensitivity derivative** calculation:

\[
R(Q(x_d), X(x_d), x_d) = 0
\]

(Discretized Residual)

\[
\frac{\partial R}{\partial Q} \cdot \left( \frac{dQ}{dx_D} \right) = - \left[ \frac{\partial R}{\partial X} \cdot \left( \frac{dX}{dx_D} \right) + \frac{\partial R}{\partial x_D} \right]
\]

[Jacobian Matrix]
Ongoing Work

- Previously, Jacobian matrix formation and application of Newton’s method are studied for *Laplace* and *Burgers’ equations*.
- The *direct 2-D Euler solver* is being developed
  - Calculation of *both numerical and analytical Jacobians* has been completed and their *accuracy* are being compared.
  - Implementation of Jacobians to *Newton’s Method* is in progress.
Results

- Error in numerical Jacobians are analyzed for
  - 15° ramp geometry
  - M=2.0 flow
  - inlet-outlet, symmetry and wall BCs
  - A 33x25 grid

- Jacobians are calculated using
  - already converged solution
  - 1st order S-W flux splitting

- Different $\epsilon$ values are considered.

- The size of Jacobian matrix is 3536x3536.
Results (Cont.)

Change Of Maximum Error with Perturbation Magnitude

Max. Error between Jacobians

Perturbation magnitude, $\varepsilon$
Future Work

- Revise numerical and analytical Jacobian calculations.
- Complete the Euler solver and compare the effects of Jacobians on the convergence of the solution.
- Apply parallel processing to the solution of the Jacobian matrix.
- Improve the solver by adapting Navier-Stokes equations using numerical Jacobians.
- Obtain sensitivity derivatives and investigate the effects of Jacobians on design optimization applications.