Appendix B

Static Aeroelastic Gradients

The Gradient Method

The static constraints can be written in general as

$$g = f(\boldsymbol{u}, \boldsymbol{v}) \tag{B.1}$$

The sensitivity of the j^{th} constraint to a change in the i^{th} design variable is given by

$$\frac{\partial g_j}{\partial v_i} = \frac{\partial f_j}{\partial v_i} + \frac{\partial f_j}{\partial u}^T \frac{\partial \boldsymbol{u}}{\partial v_i} \tag{B.2}$$

- In general the direct dependence may exist $\left(\frac{\partial f_j}{\partial v_i} \neq 0\right)$.
- If $\boldsymbol{M}$, $\boldsymbol{K}$, $\boldsymbol{B}$ are linear in $\boldsymbol{v}$ then $\frac{\partial f_j}{\partial v_i} \equiv 0$

Displacement constraints

$$\sum_{j=1}^{ndisp} A_{ij} u_j \le \delta_{i_{all}}, \quad g = \sum_{j=1}^{ndisp} \frac{A_{ij} u_j}{\delta_{i_{all}}} - 1$$

$$\frac{\partial f_j}{\partial u} = \left\{ \frac{A_i}{\delta_{i_{all}}} \right\}, \qquad \frac{\partial}{\partial v_i}(f_j) \equiv 0$$

•Displacement, Stress, Strain, and Static Aeroelastic responses gradients require $\frac{\partial u_g}{\partial v}$

•Differentiating the Static Equilibrium Equation (in the g-set)

$$K_{gg}\frac{\partial u_g}{\partial v} + M_{gg}\frac{\partial \ddot{u}_g}{\partial v} = \frac{\partial P_g}{\partial v} - \frac{\partial K_{gg}}{\partial v}u_g - \frac{\partial M_{gg}}{\partial v}\ddot{u}_g \qquad \text{(B.3)}$$

•$\frac{\partial u_g}{\partial v}, \frac{\partial \ddot{u}_g}{\partial v}$ are solved for by reduction and recovery similar to Appendix A. (Mass and stiffness need only be reduced once) gradient information is obtained by FBS)

For the present assume $\frac{\partial P_g}{\partial v}$ external loads do not depend on the design variables.

- This is true for both the steady and unsteady aerodynamic matrices currently under consideration

Therefore the RHS of (B.3)can be defined as

$$\frac{\partial \boldsymbol{R}_g}{\partial v_i} = -\frac{\partial \boldsymbol{K}_{gg}}{\partial v_i} u_g - \frac{\partial \boldsymbol{M}_{gg}}{\partial v_i} \ddot{u}_g \qquad \text{(B.4)}$$

- Reduction of the load vector to the *n-set* follows as in Appendix A

$$\frac{\partial \boldsymbol{R}_n}{\partial v_i} = \frac{\partial \bar{\boldsymbol{R}}_n}{\partial v_i} + T_{mn} \frac{\partial R_m}{\partial v_i} \qquad \text{(B.5)}$$

The single point constraints are removed by a partition of the *n-set* to give $\frac{\partial \boldsymbol{R}_n}{\partial v_i}$. Further partitioning gives the *a-set* and the *o-set*

$$\frac{\partial \boldsymbol{R}_a}{\partial v_i} = \frac{\partial \bar{\boldsymbol{R}}_a}{\partial v_i} + G_o \frac{\partial R_o}{\partial v_i} \tag{B.6}$$

Finally the partitioning to the l-set and r-set results in an equation equivalent to equation 24 in Appendix A.

CRD

$$\begin{bmatrix} \boldsymbol{K}_{ll} & \boldsymbol{K}_{lr} & \boldsymbol{M}_{ll}\boldsymbol{D} + \boldsymbol{M}_{lr} \\ 0 & 0 & \boldsymbol{m}_r \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \boldsymbol{u}_l}{\partial v_i} \\ \dfrac{\partial \boldsymbol{u}_r}{\partial v_i} \\ \dfrac{\partial \ddot{\boldsymbol{u}}_r}{\partial v_i} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \boldsymbol{R}_l}{\partial v} \\ \boldsymbol{D}^T \dfrac{\partial \boldsymbol{R}_l}{\partial v_i} + \dfrac{\partial \boldsymbol{R}_r}{\partial v_i} \end{bmatrix} \qquad \text{(B.7)}$$

From the third row of (B.7) $\dfrac{\partial \ddot{\boldsymbol{u}}_r}{\partial v_i}$ can be determined.

This result and invoking the assumption $\dfrac{\partial \boldsymbol{u}_r}{\partial v_i} \equiv 0$ enables solution of $\dfrac{\partial \boldsymbol{u}_l}{\partial v_i}$

Recovery of the *l-set* acceleration sensitivities follows as $\frac{\partial \ddot{\boldsymbol{u}}_l}{\partial v_i} = D\frac{\partial \ddot{\boldsymbol{u}}_r}{\partial v_i}$

Displacement sensitivities in the *o-set* are then obtained

$$\frac{\partial u_o}{\partial v_i} = K_{oo}^{-1}\left[\frac{\partial R_o}{\partial v_i} - [M_{oo}G_o + M_{oa}]\frac{\partial u_a}{\partial v_i}\right] + G_o\frac{\partial u_a}{\partial v_i} \qquad \text{(B.8)}$$

Equation (B.2) gives same result if done in *f-set* or *g-set*, need only recover to *f-set*. $\frac{\partial u_f}{\partial v_i}$ is obtained by merging the *o-set* and *a-set* sensitivities.

If computations are in *f-set* must reduce $\frac{\partial f}{\partial u}$ vector to *f-set*

$$\frac{\partial f_j}{\partial u_n} = \frac{\partial \bar{f}_j}{\partial u_n} + T_{mn}\frac{\partial f_j}{\partial u_m} \qquad \text{(B.9)}$$

Static Aeroelastic Sensitivities

Differentiating the static aeroelastic equation of motion in the *f-set* gives

$$K_{ff}\frac{\partial u_f}{\partial v} + M_{ff}\frac{\partial \ddot{u}_f}{\partial v} = P_f^a\frac{\partial \delta}{\partial v_i} - \frac{\partial K_{ff}}{\partial v}u_f - \frac{\partial M_{ff}}{\partial v}\ddot{u}_f \qquad \text{(B.10)}$$

The sensitivity of the orthogonality condition is also required

$$\begin{bmatrix} D & I \end{bmatrix}^T \begin{bmatrix} M_{ll} & M_{lr} \\ M_{rl} & M_{rr} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_l}{\partial v_i} \\ \dfrac{\partial u_r}{\partial v_i} \end{Bmatrix} = -\begin{bmatrix} D & I \end{bmatrix}^T \begin{bmatrix} \dfrac{\partial M_{ll}}{\partial v_i} & \dfrac{\partial M_{lr}}{\partial v_i} \\ \dfrac{\partial M_{rl}}{\partial v_i} & \dfrac{\partial M_{rr}}{\partial v_i} \end{bmatrix} \begin{Bmatrix} u_l \\ u_r \end{Bmatrix} \qquad \text{(B.11)}$$

This equation is used as a constraint when constructing

Partitioning (B.10) into the l-set and r-set using (B.11) results in

$$\begin{bmatrix} \boldsymbol{K}_{ll}^{a} & \boldsymbol{K}_{lr}^{a} & \boldsymbol{M}_{ll}\boldsymbol{D} + \boldsymbol{M}_{lr} \\ \boldsymbol{D}^{T}\boldsymbol{M}_{ll} + \boldsymbol{M}_{rl} & \boldsymbol{D}^{T}\boldsymbol{M}_{lr} + \boldsymbol{M}_{rr} & \boldsymbol{0} \\ \boldsymbol{D}^{T}\boldsymbol{K}_{ll}^{a} + \boldsymbol{K}_{rl}^{a} & \boldsymbol{D}^{T}\boldsymbol{K}_{lr}^{a} + \boldsymbol{K}_{rr}^{a} & \boldsymbol{m}_{r} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \boldsymbol{u}_{l}}{\partial v_{i}} \\ \dfrac{\partial \boldsymbol{u}_{r}}{\partial v_{i}} \\ \dfrac{\partial \ddot{\boldsymbol{u}}_{r}}{\partial v_{i}} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{P}_{l}^{a} \\ \boldsymbol{0} \\ \boldsymbol{D}^{T}\boldsymbol{P}_{l}^{a} + \boldsymbol{P}_{r}^{a} \end{Bmatrix} \delta + \begin{Bmatrix} \dfrac{\partial R_{l}}{\partial v_{i}} \\ D^{T}\dfrac{\partial R_{l}^{o}}{\partial v_{i}} + \dfrac{\partial R_{r}^{o}}{\partial v_{i}} \\ D^{T}\dfrac{\partial R_{l}}{\partial v_{i}} + \dfrac{\partial R_{r}}{\partial v_{i}} \end{Bmatrix} \qquad \text{(B.12)}$$

or

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_{1}}{\partial v_{1}} \\ \dfrac{\partial u_{2}}{\partial v_{2}} \end{Bmatrix} = \begin{Bmatrix} P_{1} \\ P_{2} \end{Bmatrix} \dfrac{\partial \delta}{\partial v_{i}} + \begin{bmatrix} \dfrac{\partial R_{1}}{\partial v_{i}} \\ \dfrac{\partial R_{2}}{\partial v_{i}} \end{bmatrix}$$

Where $\dfrac{\partial u_l}{\partial v_i}$ is recovered from

$$\frac{\partial u_l}{\partial v_i} = R_{11}^{-1}\left[P_l^a\frac{\partial\delta}{\partial v_i} + \frac{\partial R_l}{\partial v_i} - \frac{\partial R_{12}}{\partial v_i}\frac{\partial u_r}{\partial v_i} - R_{13}\frac{\partial \ddot{u}_r}{\partial v_i}\right] \qquad \text{(B.13)}$$

Equation (B.12) is the basic equation for sensitivity analysis. The unknown values (either trim parameters, δ or accelerations, $\ddot{u}_2$) have sensitivity values while those that have been fixed do not.

$$\left[K_{22} - K_{21}K_{11}^{-1}K_{12}\right]\frac{\partial u_2}{\partial v_i} = \left[P_2 - K_{21}K_{11}^{-1}P_1\right]\frac{\partial\delta}{\partial v_i} + \frac{\partial R_2}{\partial v_i} - K_{21}K_{11}^{-1}\frac{\partial R_1}{\partial v_i} \qquad \text{(B.14)}$$

Notice that ***L*** and ***R*** from the static aeroelastic equation appear here and need not be recomputed.

Partitioning and moving all unknown sensitivities on LHS yields

$$\begin{bmatrix} L_{ff} & -R_{fu} \\ L_{kf} & -R_{ku} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_{2f}}{\partial v_i} \\ \dfrac{\partial \delta_u}{\partial v_i} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial P_{2f}}{\partial v_i} \\ \dfrac{\partial P_{2k}}{\partial v_i} \end{Bmatrix} - \begin{Bmatrix} \left[K_{21} K_{11}^{-1} \dfrac{\partial P_1}{\partial v_i} \right]_f \\ \left[K_{21} K_{11}^{-1} \dfrac{\partial P_1}{\partial v_i} \right]_k \end{Bmatrix} \qquad \text{(B.15)}$$

The basic trim sensitivity analysis involves the solution of (B.15) for $\dfrac{\partial u_{2f}}{\partial v_i}, \dfrac{\partial \delta_u}{\partial v_i}$.

The displacement sensitivities are recovered by (B.13) and noting $u_1 \equiv u_r$, $u_2 \equiv \ddot{u}_r$.

The *f-set* sensitivities are determined by merging the *o-set* and the *a-set* where the *o-set* is

$$\frac{\partial u_o}{\partial v_i} = G_o^a \frac{\partial u_a}{\partial v_i} + [K_{oo}^a]^{-1} P_o^a \frac{\partial \delta}{\partial v_i}$$

Aileron Effectiveness Sensitivity

Recall aileron effectiveness definition as

$$\varepsilon_{eff} = \frac{\left(C_{l_{\delta_a}}\right)^f}{\left(C_{l_{p\frac{b}{2V}}}\right)^f}$$

where

$$\left(C_{l_{\delta_a}}\right)^f = \left[D^T P_l^a + P_r^a\right]\delta_{ail} - R_{31}u_l^{ail} - R_{32}u_r^{ail}$$

$$\left(C_{l_{p\frac{b}{2V}}}\right)^f = \left[D^T P_l^a + P_r^a\right]\delta_p - R_{31}u_l^p - R_{32}u_r^p$$

with $u_l^{ail}, u_r^{ail}, u_l^p, u_r^p$ are the pseudo-deformations for unit control surface δ_{ail} and δ_p

Sensitivities require recovery of these displacements.

The sensitivities of the stability derivatives are

$$\frac{\partial C_{l_{\delta_a}}}{\partial v_i} = -\frac{2}{qSb}\begin{bmatrix} R_{31} & R_{32} \end{bmatrix}\begin{bmatrix} \dfrac{\partial u_l^{ail}}{\partial v_i} \\ \dfrac{\partial u_r^{ail}}{\partial v_i} \end{bmatrix}$$

$$\frac{\partial C_{l_{\frac{pb}{2V}}}}{\partial v_i} = -\frac{2}{qSb}\begin{bmatrix} R_{31} & R_{32} \end{bmatrix}\begin{bmatrix} \dfrac{\partial u_l^{p}}{\partial v_i} \\ \dfrac{\partial u_r^{p}}{\partial v_i} \end{bmatrix} \qquad \text{(B.16)}$$

with $\frac{\partial u_a^{ail}}{\partial v_i}$, $\frac{\partial u_a^{p}}{\partial v_i}$ are the sensitivity of the pseudo displacements found from (B.14) as

$$\frac{\partial u_r^{ail}}{\partial v_i} = K_{11}^{-1}\frac{\partial R_1^{ail}}{\partial v_1}$$

$$\frac{\partial u_r^{p}}{\partial v_i} = K_{11}^{-1}\frac{\partial R_1^{p}}{\partial v_1} \qquad \text{(B.17)}$$

Where for aileron effectiveness (Steady roll) $\frac{\partial \ddot{u}_r}{\partial v_i}, \frac{\partial \delta}{\partial v_i} = \{0\}$.

In general for any flexible stability derivative

$$\frac{\partial F}{\partial v_i} = -\begin{bmatrix} R_{31} & R_{32} \end{bmatrix}\begin{bmatrix} \frac{\partial u_l^{param}}{\partial v_i} \\ \frac{\partial u_r^{param}}{\partial v_i} \end{bmatrix} \qquad \text{(B.18)}$$

References

1. Neill, D.J., Herendeen, D.L., Venkayya, V.B., “ASTROS Enhancements, Vol III- ASTROS Theoretical Manual”, WL-TR-95-3006.

2. Neill, D. J., Johnson E. H., Herendeen K. L., “Automated structural Optimization System (ASTROS),” AFWAL-TR-883028 Volume II-User’s Manual, April 1988.