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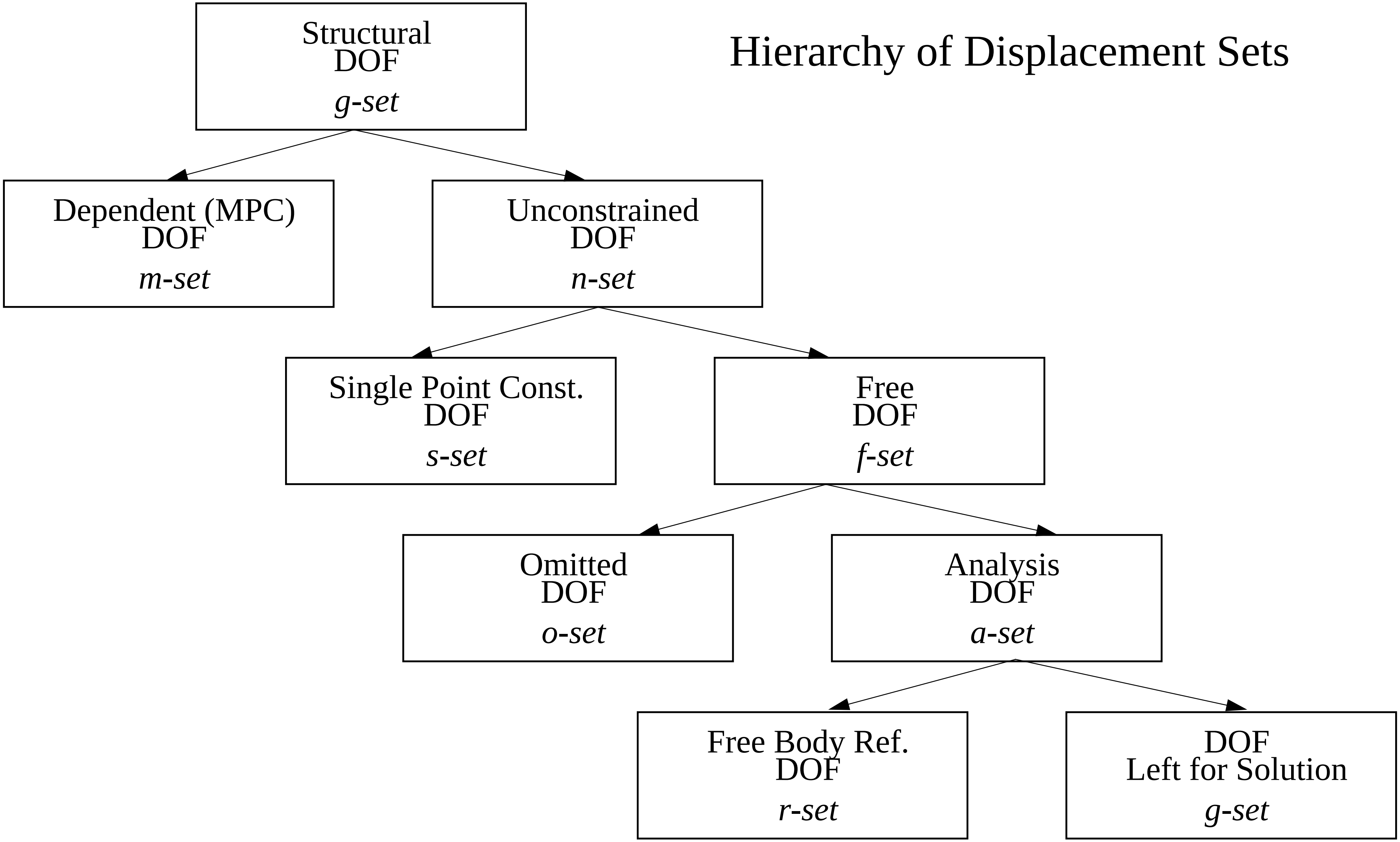
Appendix A

Static Analysis (NASTRAN/ASTROS)

Hierarchy of Displacement Sets



The equations of motion in the global or *g-set*

$$\boldsymbol{K}_{\boldsymbol{gg}}\boldsymbol{u}_{\boldsymbol{g}} + \boldsymbol{M}_{\boldsymbol{gg}}\ddot{\boldsymbol{u}}_{\boldsymbol{g}} = \boldsymbol{P}_{\boldsymbol{g}} \tag{A.1}$$

The 1st partition is for the rigid elements and MPC. Assuming that the relation between the *n-set* (Independent set) and the *m-set* (dependent set) is

$$\boldsymbol{u}_{\boldsymbol{m}} = \boldsymbol{T}_{\boldsymbol{mn}}\boldsymbol{u}_{\boldsymbol{n}},\ \ddot{\boldsymbol{u}}_{\boldsymbol{m}} = \boldsymbol{T}_{\boldsymbol{mn}}\ddot{\boldsymbol{u}} \tag{A.2}$$

The work performed by the forces introduced due to the MPC must be zero. That is

$$\boldsymbol{c}_{\boldsymbol{g}} = \begin{bmatrix} -\boldsymbol{T}^{\boldsymbol{T}}_{\boldsymbol{mn}} \\ \boldsymbol{I} \end{bmatrix} q_m \equiv 0 \tag{A.3}$$

Substituting (A.2) and (A.3) into (A.1) and partitioning yields

$$\begin{bmatrix} \overline{\boldsymbol{K}}_{nn} & \boldsymbol{K}_{nm} & \boldsymbol{T}^{T}_{mn} \\ \boldsymbol{K}_{mn} & \boldsymbol{K}_{mm} & -\boldsymbol{I} \\ \boldsymbol{T}_{mn} & -\boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{u}_n \\ \boldsymbol{u}_m \\ \boldsymbol{q}_m \end{Bmatrix} + \begin{bmatrix} \overline{\boldsymbol{M}}_{nn} & \boldsymbol{M}_{nm} \\ \boldsymbol{M}_{mn} & \boldsymbol{M}_{mm} \\ \boldsymbol{T}_{mn} & -\boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{u}}_n \\ \ddot{\boldsymbol{u}}_m \end{Bmatrix} = \begin{Bmatrix} \overline{\boldsymbol{P}}_n \\ \boldsymbol{P}_m \\ \boldsymbol{0} \end{Bmatrix} \tag{A.4}$$

Where the bar over certain terms refers to the elements in the partitions of g-size matrix before reduction to the *n-set*. This notation is used throughout the derivation.

Expanding and Substituting (A.2) for $\boldsymbol{u}_m$, $\ddot{\boldsymbol{u}}_m$ yields 3 equations

$$\begin{aligned} \bar{K}_{nn}u_n + K_{nm}T_{mn}u_n + T^T_{mn}q_m + \bar{M}_{nn}\ddot{u}_n + M_{nm}T_{mn}\ddot{u}_n &= \bar{P}_n \\ K_{mn}u_n + K_{mm}T_{mn}u_n - Iq_m + M_{mn}\ddot{u} + M_{mm}T_{mn}\ddot{u} &= P_m \\ T_{mn}u_n - IT_{mn}u_n + T_{mn}\ddot{u} - IT_{mn}\ddot{u}_n &= 0 \end{aligned} \quad (A.5)$$

Solving the second equation in (A.5) for q_m and substituting this into the first equation in (A.5) yields one equation in terms of u_n, $\ddot{u}_n$ in the form.

$$\left\{\bar{K}_{nn} + K_{nm}T_{mn} + T^T_{mn}[K_{mn} + K_{mm}T_{mn}]\right\}u_n + \quad (A.6)$$

$$\left\{\bar{M}_{nn} + M_{nm}T_{mn} + T^T_{mn}[M_{mn} + M_{mm}T_{mn}]\right\}\ddot{u} = \bar{P}_n + T^T_{mn}P_m$$

or

$$\boldsymbol{K}_{nn}\boldsymbol{u}_n + \boldsymbol{M}_{nn}\ddot{\boldsymbol{u}}_{nn} = \boldsymbol{P}_n \quad (A.7)$$

Where

$$K_{nn} = \bar{K}_{nn} + K_{nm}T_{mn} + T_{mn}^T[K_{mn} + K_{mm}T_{mn}]$$

$$M_{mm} = \bar{M}_{nn} + M_{nm}T_{mn} + T_{mn}^T[M_{mn} + M_{mm}T_{mn}]$$

$$P_n = \bar{P}_n + T_{mn}^T P_m$$

Now reduce the *n-set* based on Single Point Constraints (SPC) to the *s-set* and the *f-set*

$$u_s = Y_s \qquad \text{(A.8)}$$

From equation (A.6) (dropping subscripts) we partition into the *s-set* and *f-set* as

$$\begin{bmatrix} \boldsymbol{K}_{ff} & \boldsymbol{K}_{fs} \\ \boldsymbol{K}_{sf} & \boldsymbol{K}_{ss} \end{bmatrix} \begin{Bmatrix} \boldsymbol{u}_f \\ \boldsymbol{Y}_s \end{Bmatrix} + \begin{bmatrix} \boldsymbol{M}_{ff} & \boldsymbol{M}_{fs} \\ \boldsymbol{M}_{sf} & \boldsymbol{M}_{ss} \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{u}}_f \\ \boldsymbol{0} \end{Bmatrix} = \begin{Bmatrix} \bar{\boldsymbol{P}}_f \\ \boldsymbol{P}_s \end{Bmatrix} \qquad \text{(A.9)}$$

Expanding gives

$$\begin{aligned} K_{ff}u_f + K_{fs}Y_s + M_{ff}\ddot{u}_f &= \bar{P}_f \\ K_{sf}u_f + K_{ss}Y_s + M_{sf}\ddot{u}_f &= P_s \end{aligned} \qquad \text{(A.10)}$$

The first of equation (A.10) can be rearranged to give

$$K_{ff}u_f + M_{ff}\ddot{u}_f = \bar{P}_f - K_{fs}Y_s \tag{A.11}$$

or

$$\boldsymbol{K}_{ff}\boldsymbol{u}_f + \boldsymbol{M}_{ff}\ddot{\boldsymbol{u}}_f = \boldsymbol{P}_f \tag{A.12}$$

with

$$P_f = \bar{P}_f - K_{fs}Y_s \tag{A.13}$$

Reduction from the *f-set* to the *a-set* and *o-set* through Guyan Reduction.

$$\begin{bmatrix} \bar{\boldsymbol{K}}_{aa} & \boldsymbol{K}_{ao} \\ \boldsymbol{K}_{oa} & \boldsymbol{K}_{oo} \end{bmatrix} \begin{Bmatrix} \boldsymbol{u}_a \\ \boldsymbol{u}_o \end{Bmatrix} + \begin{bmatrix} \bar{\boldsymbol{M}}_{aa} & \boldsymbol{M}_{ao} \\ \boldsymbol{M}_{oa} & \boldsymbol{M}_{oo} \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{u}}_a \\ \ddot{\boldsymbol{u}}_o \end{Bmatrix} = \begin{Bmatrix} \bar{\boldsymbol{P}}_a \\ \boldsymbol{P}_o \end{Bmatrix} \tag{A.14}$$

Expanding yields

$$\begin{aligned} \bar{K}_{aa}u_a + K_{ao}u_o + \bar{M}_{aa}\ddot{u}_a + M_{ao}u_o &= \bar{P}_a \\ K_{oa}u_a + K_{oo}u_o + M_{oo}\ddot{u} &= P_o \end{aligned} \tag{A.15}$$

From the second equation in equation (A.15) solve for u_o as

$$u_o = K_{oo}^{-1}[P_o - K_{oa}u_a - M_{oa}\ddot{u}_a - M_{oo}\ddot{u}_a] \tag{A.16}$$

Now in a manner consistent with Guyan reduction, the mass matrix is reduced using a static condensation transformation of the mass matrix to relate the omitted and retained degrees of freedom:

$$\ddot{u}_o = -[K_{oo}^{-1} K_{oa}]\ddot{u}_a = G_o \ddot{u}_a \qquad \text{(A.17)}$$

Now substituting equations (A.16) and (A.17) into the first equation of (A.15) and rearranging yields

$$[\bar{K}_{aa} + K_{ao}G_o]u_a + [\bar{M}_{aa} + M_{ao}G_o + G^T M_{oa} + G_o^T M_{oo}G_o]\ddot{u}_a = \bar{P}_a + G_o^T P_o \qquad \text{(A.18)}$$

or

$$\boldsymbol{K}_{aa}\boldsymbol{u}_a + \boldsymbol{M}_{aa}\ddot{\boldsymbol{u}}_a = \boldsymbol{P}_a$$

with

$$\boldsymbol{K}_{aa} = \bar{K}_{aa} + K_{ao}G_o$$

$$\boldsymbol{M}_{aa} = \bar{M}_{aa} + M_{ao}G_o + G_o^T M_{oa} + G_o^T M_{oo}G_o$$

$$\boldsymbol{P}_a = \bar{P}_a + G_o^T P_o$$

With the matrices in the *a-set* the final partition is to the *l-set* and *r-set*.

CRD

$$\begin{bmatrix} \boldsymbol{K}_{ll} & \boldsymbol{K}_{lr} \\ \boldsymbol{K}_{rl} & \bar{\boldsymbol{K}}_{rr} \end{bmatrix} \begin{Bmatrix} \boldsymbol{u}_l \\ \boldsymbol{u}_r \end{Bmatrix} + \begin{bmatrix} \boldsymbol{M}_{ll} & \boldsymbol{M}_{lr} \\ \boldsymbol{M}_{rl} & \bar{\boldsymbol{M}}_{rr} \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{u}}_l \\ \ddot{\boldsymbol{u}}_r \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{P}_l \\ \bar{\boldsymbol{P}}_r \end{Bmatrix} \quad \text{(A.19)}$$

The r-set contains degrees of freedom equal in number to the number of rigid body modes in the structure. The r-set displacements are arbitrarily set to zero($u_a = 0$). This does not mean however that $\ddot{u}_a = 0$. Consistent with the methodology in both ASTROS and NAS-TRAN solving equation (A.19) begins with the determination of the rigid body mode shapes. They are determined by solving for the displacements of an unloaded structure using the stiffness matrix as

$$u_l = -K_{ll}^{-1} K_{lr} u_r \quad \text{(A.20)}$$

or

$$\begin{Bmatrix} u_l \\ u_r \end{Bmatrix} = \begin{bmatrix} D \\ I \end{bmatrix} \{u_r\} \quad \text{(A.21)}$$

with

$$D = -K_{ll}^{-1} K_{lr} u_r \tag{A.22}$$

designated the *rigid body transformation* matrix. Also, assuming uniform acceleration as a rigid body (here the accelerations include only rigid body motions) the $\ddot{u}_l$ can be expressed i a similar manner

$$\begin{Bmatrix} \ddot{u}_l \\ \ddot{u}_r \end{Bmatrix} = \begin{bmatrix} D \\ I \end{bmatrix} \{\ddot{u}_r\} \tag{A.23}$$

With equations (A.21) and (A.22) the transformation of (A.19) to the *r-set* can be performed as

$$\lfloor D^T \quad I \rfloor \begin{bmatrix} K_{ll} & K_{lr} \\ K_{rl} & \bar{K}_{rr} \end{bmatrix} \begin{bmatrix} D \\ I \end{bmatrix} \{u_r\} + \lfloor D^T \quad I \rfloor \begin{bmatrix} M_{ll} & M_{lr} \\ M_{rl} & \bar{M}_{rr} \end{bmatrix} \begin{bmatrix} D \\ I \end{bmatrix} \{\ddot{u}\} = \lfloor D^T \quad I \rfloor \begin{Bmatrix} P_l \\ \bar{P}_r \end{Bmatrix} \tag{A.24}$$

Expanding yields

$$[D^T K_{ll} D + D^T K_{lr} + K_{rl} D + \bar{K}_{rr}]\{u_r\} + [D^T M_{ll} D + D^T M_{lr} + \bar{M}_{rr}]\{\ddot{u}_r\} \tag{A.25}$$
$$= D^T P_l + \bar{P}_r$$

or

$$\boldsymbol{K}_{rr}\boldsymbol{u}_r + \boldsymbol{M}_{rr}\ddot{\boldsymbol{u}}_r = \boldsymbol{P}_r \tag{A.26}$$

with after substitution for D, D^T in K_{rr}

$$\boldsymbol{K}_{rr} = \bar{K}_{rr} - K_{rl}K_{ll}^{-1}K_{lr}$$
$$\boldsymbol{M}_{rr} = \boldsymbol{m}_r = D^T M_{ll} D + D^T M_{lr} + \bar{M}_{rr} \tag{A.27}$$
$$\boldsymbol{P}_r = D^T P_l + \bar{P}_r$$

where $\boldsymbol{m}_r$ is often referred to as the *rigid body mass* matrix.

Now invoking the initial assumption $\boldsymbol{u}_r = 0$ equation (A.26) becomes

$$\boldsymbol{M}_{rr}\ddot{\boldsymbol{u}}_r = \boldsymbol{P}_r \tag{A.28}$$

and the rigid body accelerations due to the applied loads can be computed as

$$\ddot{\boldsymbol{u}}_r = \boldsymbol{M}_{rr}^{-1}\boldsymbol{P}_r \tag{A.29}$$

With $\ddot{u}_r$ and $\ddot{u}_l$ can be found from equation (A.23) to be

$$\ddot{u}_l = D\ddot{u}_r \tag{A.30}$$

Substituting this result into the first equation in (A.19) and again recalling that $u_r = 0$ the u_l displacements can be solved for as

$$\boldsymbol{u}_l = K_{ll}^{-1}[P_l - M_{ll}D\ddot{u}_r - M_{lr}\ddot{u}_r] \tag{A.31}$$

where it is worth noting that $-M_{ll}D\ddot{u}_r - M_{lr}\ddot{u}_r$ is the modification of the load vector due to rigid body inertial effects. u_l are the displacements in the *l-set* including *inertial relief effects.* These deformations are relative to the support point

Recover of the Displacements to the *g-set*

Once u_l has been determined the displacement and accelerations in the a-set are found by a simple merge operation as

$$u_a = \begin{Bmatrix} u_l \\ u_r \end{Bmatrix} = \begin{Bmatrix} u_l \\ 0 \end{Bmatrix}$$

$$\ddot{u}_a = \begin{Bmatrix} \ddot{u}_l \\ \ddot{u}_r \end{Bmatrix} \tag{A.32}$$

To recover the *o-set* use equations (A.16) and (A.17) which yields

$$u_o = K_{oo}^{-1}[P_o - K_{oa}u_a - M_{oa}\ddot{u}_a - M_{oo}\ddot{u}_a]$$

$$\ddot{u}_o = -[K_{oo}^{-1}K_{oa}]\ddot{u}_a = G_o\ddot{u}_a \tag{A.33}$$

Where $-M_{oa}\ddot{u}_a - M_{oo}\ddot{u}_a = P_o^i$ can be identified as the inertial loads. Also noting that $K_{oo}^{-1}K_{oa} = G_o$ the equation (A.33) can be written as

$$u_o = K_{oo}^{-1}[P_o + P_o^i] + G_o u_a$$
$$\ddot{u}_o = -[K_{oo}^{-1} K_{oa}]\ddot{u}_a = G_o \ddot{u}_a \qquad \text{(A.34)}$$

To recover to the *f-set* simply merge the *a-set* and the *o-set* to

$$u_f = \begin{Bmatrix} u_a \\ u_o \end{Bmatrix}$$
$$\ddot{u}_f = \begin{Bmatrix} \ddot{u}_a \\ \ddot{u}_o \end{Bmatrix} \qquad \text{(A.35)}$$

For the *n-set* recovery we merge the *f-set* and *a-set*. The *s-set* accelerations are zero and the displacements are specified in Y_s. Therefore the *n-set* is

$$u_n = \begin{Bmatrix} u_f \\ Y_s \end{Bmatrix} \tag{A.36}$$

$$\ddot{u}_n = \begin{Bmatrix} \ddot{u}_f \\ 0 \end{Bmatrix}$$

For recovery of the m-set dependent displacements and accelerations use equation (A.2) which gives

$$\begin{aligned} u_m &= T_{mn} u_n \\ \ddot{u}_m &= T_{mn} \ddot{u}_n \end{aligned} \tag{A.37}$$

Using the second equation in equation (A.5) the MPC forces q_m can be obtained as

$$q_m = K_{mn} u_n + K_{mm} T_{mn} u_n + M_{nm} \ddot{u} + M_{mm} T_{mn} \ddot{u}_n \tag{A.38}$$

Finally, to get the *g-set* displacements, merge the *n-set* and the *m-set* to obtain

CRD

$$u_g = \begin{Bmatrix} u_n \\ u_m \end{Bmatrix}$$

$$\ddot{u}_g = \begin{Bmatrix} \ddot{u}_n \\ \ddot{u}_m \end{Bmatrix} \tag{A.39}$$

As a final note. When the n-set was reduced to the *f-set* and the *s-set* the SPC forces were not included in equation (A.9) If the SPC forces are of interest equation (A.9) can be recast as

$$\begin{bmatrix} \boldsymbol{K}_{ff} & \boldsymbol{K}_{fs} & \boldsymbol{0} \\ \boldsymbol{K}_{sf} & \boldsymbol{K}_{ss} & -\boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \boldsymbol{u}_f \\ \boldsymbol{u}_s \\ \boldsymbol{q}_s \end{Bmatrix} + \begin{bmatrix} \boldsymbol{M}_{ff} & \boldsymbol{M}_{fs} & \boldsymbol{0} \\ \boldsymbol{M}_{sf} & \boldsymbol{M}_{ss} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{u}}_f \\ \boldsymbol{0} \\ \boldsymbol{0} \end{Bmatrix} = \begin{Bmatrix} \bar{\boldsymbol{P}}_f \\ \boldsymbol{P}_s \\ \boldsymbol{Y}_s \end{Bmatrix} \tag{A.40}$$

Using the second row of equation (A.40) the SPC forces q_s can be solved for as

$$q_s = -P_s + K_{sf}u_f + K_{ss}u_s + M_{sf}\ddot{u}_f \tag{A.41}$$