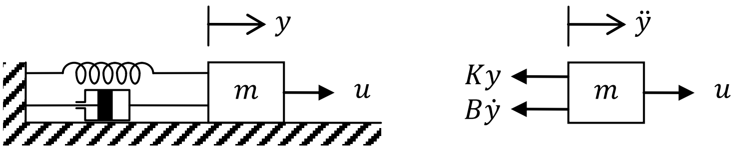
Spring-Mass-Damper System Example

Consider the following spring-mass system:



m: mass in kg
y: position in m
u: control input in N
K: linear spring constant in N/m
B: damping coefficient in Ns/m

Motion of the mass under the applied control, spring, and damping forces is governed by the following second order linear ordinary differential equation (ODE):

$$m\ddot{y} + B\dot{y} + Ky = u \tag{1}$$

Taking the Laplace transform of (2) yields the following transfer function from the input u to the output y:

$$G(s) = \frac{y(s)}{u(s)} = \frac{1}{ms^2 + Bs + K} \tag{2}$$

Setting the denominator of the transfer function to zero yields the **characteristic equation** of the above system:

$$ms^2 + Bs + K = 0 \tag{3}$$

Roots of the characteristic equation are called the **poles** of the system and give important information about the dynamic characteristics of the system. Poles of a system can be

real (imaginary part equal to zero): $p_1 = a_1$, $p_2 = a_2$

or **complex** (nonzero imaginary part): $p_1 = a + bi$, $p_2 = a - bi$

Response to Initial Conditions

Response of a second order system to nonzero initial conditions with no control input can be obtained by solving (2) with $u = 0$.

For **real poles** the solution is

$$y(t) = c_1 e^{a_1 t} + c_2 e^{a_2 t} \quad \text{with} \quad \begin{aligned} c_1 &= \tfrac{1}{a_2 - a_1}\left(a_2 y(0) - y'(0)\right) \\ c_2 &= \tfrac{1}{a_2 - a_1}\left(-a_1 y(0) + y'(0)\right) \end{aligned}$$

where $y(0)$ and $y'(0)$ are the initial position and velocity.

For **complex poles** the solution is

$$y(t) = e^{at}(c_1 \cos bt + c_2 \sin bt) \quad \text{with} \quad \begin{aligned} c_1 &= y(0) \\ c_2 &= -\tfrac{a}{b} y(0) + \tfrac{1}{b} y'(0) \end{aligned}$$

For the mass-spring system poles are obtained from (3) as

$$p_{1,2} = \frac{-B \pm \sqrt{B^2 - 4mK}}{2m}$$

Complex poles (oscillatory response) occur for $|B| < \sqrt{4mK}$.

Exercise: Important if you are not familiar with Matlab! Study the response of the mass-spring system to various initial conditions using the Matlab file SpringMassInit.m. Observe the open-loop pole locations and system response for

a) Keep $m = 0.1$ and $K = 1$ constant and run the file for $B = \{0.05, 0.1, 0.2, 0.4, 0.7, 0, -0.01\}$.
b) Keep $m = 0.1$ and $B = 0.05$ constant and run the file for $K = \{0.2, 0.5, 1, 2.5, 0, -0.01\}$.

Response to Open-Loop Commands

The above analysis assumed zero control command to the system. Response of the system to open-loop control inputs can be studied by solving (2) with the given $u(t)$. Another way is to insert the Laplace transform of the input signal ($u(s)$) into (2) and get the output response in Laplace domain. The response in time domain can then be obtained by taking the inverse Laplace transform. Usually partial fraction expansion and Laplace tables are used for Laplace inversions. For simple wave forms (impulse, step, ramp, sinusoids, etc.) analytical expressions for the output response may be obtained. For more complex input signals output responses can be obtained by numerical solution.

Exercise: Study the response of the mass-spring system to various open-loop commands by using the Simulink file SpringMass.mdl. First run the initialization file SpringMassInit. Then replace the input block “u” with the following blocks and run the simulation:

a) “Step” block in Simulink Library/Sources,
b) “Ramp” block in Simulink Library/Sources,
c) “Sine Wave” block in Simulink Library/Sources. Set the frequency of the Sine Wave to 10 rad/s.