1 Introduction

The instability and atomization problems of liquid jets have attracted the interest of researchers for a long time since they have applications in many industrial processes and machines. Most of these processes are related to the disintegration of a jet into small droplets after being discharged into a stagnant gas. Fuel injection and combustion in internal combustion engines or gas generators, droplets after being discharged into a stagnant gas. Fuel injection and combustion in internal combustion engines or gas generators, and combustion problems in jet engines or internal combustion engines, including liquid-gas interaction and nozzle geometry effects. Their results indicate that as the velocity profile of the jet stabilizes the flow in the atomization regime, while the density stratification and electric charges destabilize it. Additionally, a fully developed flow is more stable compared to an underdeveloped one. For the Rayleigh regime, both the surface tension and electric charges destabilize the flow.

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The instability characteristics of a liquid jet discharging from a nozzle into a stagnant gas are investigated using the linear stability theory. Starting with the equations of motion for incompressible, inviscid, axisymmetric flows in cylindrical coordinates, a dispersion relation is obtained, where the amplification factor of the disturbance is related to its wave number. The parameters of the problem are the laminar velocity profile shape parameter, surface tension, fluid densities, and electrical charge of the liquid jet. The dispersion relation is numerically solved as a function of the wave number. The growth of instabilities occurs in two modes, the Rayleigh and atomization modes. For \( r \text{We} < 1 \) (where \( \text{We} \) represents the Weber number and \( r \) represents the gas-to-liquid density ratio) corresponds to a Rayleigh or long wave instability, where atomization does not occur. On the contrary, for \( \text{We} \gg 1 \) the waves at the liquid-gas interface are shorter and when they reach a threshold amplitude the jet breaks down or atomizes. The surface tension stabilizes the flow in the atomization regime, while the density stratification and electric charges destabilize it. Additionally, a fully developed flow is more stable compared to an underdeveloped one. For the Rayleigh regime, both the surface tension and electric charges destabilize the flow. [DOI: 10.1115/1.4007157]
In a theoretical analysis, Artana et al. [8] have developed a temporal linear stability analysis of a circular electrified jet flowing inside a cylindrical coaxial electrode. The most important finding of the study is that the electrification acts on the stability of a jet in a different way, depending on whether the surface is considered equipotential or not. For the equipotential case, an increase in the velocity or in the electric field or a decrease in the surface tension destabilizes a jet. For the case of frozen charges on the jet surface, the effect of the electric field is opposed to the equipotential case.

In an experimental effort, Artana et al. [9] deal with modifications of the breakup phenomena of a high-velocity jet when stressed by an electric field. The results indicate that the electric field increases the radial/axial velocity component ratio and promotes the earlier detachment of droplets from the jet. However, no change in the droplet sizes is observed.

In a theoretical and experimental study, Priol et al. [10] studied the stability of diesel oil jets, taking the effects of inertia, surface tension, viscous, aerodynamic, and electric forces into account. It was concluded from the theoretical analysis that the atomization of the jet is enhanced with the injection of electrical charges and the experimental results confirmed this finding.

The flow geometry investigated in this study is shown in Fig. 1. As can be seen from the figure, an axisymmetric laminar liquid jet discharges into a stagnant gas through a nozzle. While \( r^* \) is the radial direction, \( z^* \) is the axial or the streamwise direction. As can be seen from Fig. 1, after leaving the nozzle the jet flow remains stable until a certain distance from the nozzle exit. However, after a certain distance two dimensional disturbances start to occur which, upon amplification, result in either the Rayleigh mode or atomization. The aim of the study is to parametrically investigate the stability characteristics of this flow system using the linear stability theory. In the approach that is adopted here, the densities of the liquid and the gas, surface tension, and the laminar velocity profile shape of the liquid jet are the parameters of the problem. Additionally, an electrically charged liquid jet can also be taken into account. In this study, both the liquid and the gas are considered to be inviscid. Although this approach may seem to be too restrictive, it is appropriate for the objectives of this study because a parametric study using a linear approach is attempted.

The analysis starts with the equations of motion for incompressible, inviscid, axisymmetric flows in cylindrical coordinates. By using the small disturbance theory followed by a normal mode analysis, a dispersion relation is obtained for the complex wave velocity as a function of the disturbance wave number, where the laminar velocity profile shape parameter, surface tension, density stratification, and electrical charge parameters are the parameters of the problem. The solution of the dispersion relation using a numerical method yields the wavelengths, wave speeds, and the frequencies of the interfacial disturbances. These instability properties influence the atomization characteristics of jets, according to Reitz and Bracco [1], Ibrahim and Marshall [2], and Turnbull [7]. In this study, the approach outlined by Ibrahim and Marshall [2] is adopted. The effect of an electrically charged jet is modeled by applying the approach outlined by Lasheras and Hopfinger [3] to the flow treated here.

2 Problem Description and Formulation

In the following equations, the subscripts 1 and 2 denote the liquid and gas phases, respectively. The radius of the jet is \( a^* \). While \( r^* = a^* \) is the undisturbed liquid-gas interface, \( r^* = a^* + \eta^* \) is the disturbed interface, where \( \eta^* \) is the amplitude of the disturbance. Asterisks (*) denote dimensional quantities. The axial velocity profile of the liquid jet is denoted by \( U_{2z}^*(r^*) \), while the gas is stagnant, i.e., \( U_{2z}^* = 0 \). As can be seen from Fig. 1, after leaving the nozzle the jet flow remains stable until a certain distance from the nozzle exit. However, when a threshold streamwise distance is reached, two dimensional disturbances start to occur. The disturbances amplify in magnitude in the downstream direction. The amplification of the disturbances end up in one of the two following alternatives:

- Rayleigh regime: In this instability mode the waves are relatively long and the jet preserves its integrity for a long distance in the downstream direction due to the surface tension. After a threshold distance from the nozzle exit is reached, the waves break down, however atomization does not occur.
- Atomization regime: In this instability mode the waves are short. When the disturbance wave amplitude reaches a threshold value, the liquid jet breaks down to small droplets due to the aerodynamic effect of the gas. As the magnitude of the \( rWe \) term increases, the jet breaks down into smaller droplets.

2.1 Dispersion Relation and Solution Method. Equations of motion for an incompressible, inviscid, axisymmetric flow in cylindrical coordinates are employed.

The momentum equation in the \( r^* \)-direction is

\[
\frac{\partial \bar{u}_r}{\partial r^*} + \bar{u}_r \frac{\partial \bar{u}_r}{\partial r} + \bar{u}_z \frac{\partial \bar{u}_r}{\partial z} = -\frac{1}{\rho^*} \frac{\partial p^*}{\partial r^*} \tag{1}
\]

The momentum equation in the \( z^* \)-direction is

\[
\frac{\partial \bar{u}_z}{\partial z^*} + \bar{u}_r \frac{\partial \bar{u}_z}{\partial r} + \bar{u}_z \frac{\partial \bar{u}_z}{\partial z} = -\frac{1}{\rho^*} \frac{\partial p^*}{\partial z^*} \tag{2}
\]
The continuity equation is
\[ \frac{\partial u'_i}{\partial r} + \frac{u'_i}{r} + \frac{\partial u'_z}{\partial z} = 0 \] (3)

Following the linear stability theory, the flow is decomposed into a steady mean component and an unsteady perturbation
\[ u'_i = U'_i(r', z') + \hat{u}'(r', z', t') \]
\[ u'_z = U'_z(r', z') + \hat{u}'(r', z', t') \] (4)
\[ p' = P'(r', z') + \hat{p}'(r', z', t') \] (5)

At this stage, the following assumptions are made:
1. Perturbation components are much smaller in magnitude compared to mean components; therefore, quadratic terms are neglected.
2. Mean flow satisfies the equations of motion.
3. The flow can be regarded as parallel, i.e., \( U'_i = U'_i(r') \) only and \( U'_z = 0 \).

When Eqs. (4)–(6) are substituted into Eqs. (1)–(3), a linear set of stability equations are obtained as follows:
\[ \frac{\partial \hat{u}_1}{\partial t} + U'_1 \frac{\partial \hat{u}_1}{\partial z} = - \frac{1}{\rho'_1} \frac{\partial \hat{p}_1}{\partial r} \] (7)
\[ \frac{\partial \hat{u}_2}{\partial t} + \hat{u}_2 \frac{dU'_1}{dr} + U'_1 \frac{\partial \hat{u}_2}{\partial z} = - \frac{1}{\rho'_2} \frac{\partial \hat{p}_2}{\partial r} \] (8)
\[ \frac{\partial \hat{u}_3}{\partial t} + \frac{\hat{u}_3}{r} + \frac{\partial \hat{u}_2}{\partial z} = 0 \] (9)

At this stage, it is appropriate to nondimensionalize the equations, where all lengths are reduced by the jet radius \( a' \), velocities by the jet centerline velocity \( U'_{1,\text{max}} \), pressures by \( \rho'_1 U'_{1,\text{max}}^2 \), and time \( t' \) by \( a'/U'_{1,\text{max}} \). In dimensional coordinates, the unperturbed liquid-gas interface is at \( r = 1 \), while the perturbed interface is at \( r = 1 + \eta \). Accordingly, Eqs. (7)–(9) can be rewritten for the liquid jet as follows:
\[ \frac{\partial \hat{u}_1}{\partial t} + U'_1 \frac{\partial \hat{u}_1}{\partial z} = - \frac{1}{\rho'_1} \frac{\partial \hat{p}_1}{\partial r} \] (10)
\[ \frac{\partial \hat{u}_2}{\partial t} + \hat{u}_2 \frac{dU'_1}{dr} + U'_1 \frac{\partial \hat{u}_2}{\partial z} = - \frac{1}{\rho'_2} \frac{\partial \hat{p}_2}{\partial r} \] (11)
\[ \frac{\partial \hat{u}_3}{\partial t} + \frac{\hat{u}_3}{r} + \frac{\partial \hat{u}_2}{\partial z} = 0 \] (12)

In the same manner, the dimensionless equations for the gas layer are written as
\[ \frac{\partial \hat{u}_2}{\partial t} = - \frac{\rho'_2}{\rho'_1} \frac{\partial \hat{p}_2}{\partial r} \] (13)
\[ \frac{\partial \hat{u}_3}{\partial t} = - \frac{\rho'_2}{\rho'_1} \frac{\partial \hat{p}_2}{\partial r} \] (14)
\[ \frac{\partial \hat{u}_2}{\partial t} + \frac{\hat{u}_2}{r} + \frac{\partial \hat{u}_3}{\partial z} = 0 \] (15)

In the next step, a stream function is defined for the perturbation velocities for the liquid \( (j = 1) \) and gas \( (j = 2) \) layers
\[ \Psi_j(r, z, t) = \psi_j(r) e^{i(\alpha - ct)} \] (16)

In the preceding expression, \( \psi_j(r) \) represents the disturbance amplitude function while, according to the temporal amplification theory, \( \alpha \) represents the disturbance wave number and \( c = c_r + ic_i \) represents the complex wave velocity [11]. The real part of the complex wave velocity represents the phase speed, while the complex part is the amplification factor. The flow is stable when \( c_r < 0 \), unstable when \( c_r > 0 \), or neutrally stable when \( c_r = 0 \). Perturbation velocities are expressed in terms of the perturbation stream function given in Eq. (16) as follows:
\[ \hat{u}_1 = \frac{1}{r} \frac{\partial \Psi_j}{\partial z}, \quad \hat{u}_2 = - \frac{1}{r} \frac{\partial \Psi_j}{\partial r} \] (17)

Notice that the perturbation velocity components satisfy the continuity equations, namely, Eqs. (12) and (15). In the next step, the pressure terms need to be eliminated between equation pairs (10) and (11) and (13) and (14) by cross-differentiation and adding the equation pairs. After rearranging the terms, the following equation is obtained for the liquid jet:
\[ \frac{\partial}{\partial t} \left( \frac{\partial \hat{u}_1}{\partial z} - \frac{\partial \hat{u}_2}{\partial r} \right) = 0 \] (18)

The equation for the gas layer is similarly obtained, after remembering that \( U'_{1,\text{gas}} = 0 \)
\[ \frac{\partial}{\partial t} \left( \frac{\partial \hat{u}_2}{\partial z} - \frac{\partial \hat{u}_3}{\partial r} \right) = 0 \] (19)

Expressing the perturbation velocity components in terms of the perturbation stream function, as defined in Eq. (17), yields the following equations for the liquid and gas layers:
\[ \text{iz} \{ U'_{1,\text{max}} - c \} \left( r \frac{d^2 \Psi_1}{dr^2} - \frac{d \Psi_1}{dr} - r^2 \Psi_1 \right) - \text{iz} \left( r \frac{d^2 U'_{1,\text{max}}}{dr^2} \right) \Psi_1 = 0 \] (20)
\[ \text{iz} \left( r \frac{d^2 \Psi_2}{dr^2} - \frac{d \Psi_2}{dr} - r^2 \Psi_2 \right) = 0 \] (21)

At this stage, a velocity profile needs to be determined for the laminar liquid jet flow [2]
\[ U'_1(r) = 1 - br^2 \] (22)

In the preceding expression, \( b \) is the velocity profile parameter and varies between 0 for a uniform profile and 1 for a Hagen–Poiseuille profile; see Fig. 2. In a fully developed laminar axisymmetric pipe flow, the velocity profile is the parabolic Hagen–Poiseuille profile. The distance required for a flow to be fully developed increases with an increasing flow velocity and a decreasing nozzle length. In other words, the velocity profile deviates from parabolic for a faster flow and/or a shorter nozzle. The velocity profile parameter allows for the investigation of the nozzle length and jet velocity as an additional parameter in the stability problem. When Eq. (22) is used in Eq. (20), an identical expression is obtained for both the liquid and gas layers
\[ \frac{d^2 \Psi_j}{dr^2} - \frac{1}{r} \frac{d \Psi_j}{dr} - r^2 \Psi_j = 0 \] (23)

This is a Bessel equation and its general solution is as follows [12]:
\[ \Psi_j = c_1 J_1(ax) + c_2 K_1(ax) \] (24)
In the preceding expression, $I_0$ and $K_1$ are the modified first order Bessel functions of the first and second kind, respectively. In order to determine the integration constants $c_1$ and $c_2$, linear interface conditions must be defined and employed:

1. The fluid particles at the liquid-gas interface ($r = 1$) remain there

   For the liquid: $\hat{u}_1 = \frac{\partial \eta}{\partial r} + U_z \frac{\partial \eta}{\partial z}$  \hspace{1cm} (25)

   For the gas: $\hat{u}_2 = \frac{\partial \eta}{\partial r}$  \hspace{1cm} (26)

2. The pressure difference between the liquid and gas layers is balanced by surface tension and stresses due to the electric charge of the liquid jet. This principle is expressed in dimensional form as follows [11]:

   $$\hat{p}_1 = \hat{p}_2 - \frac{\sigma^*}{\varepsilon_0^*} \left( \eta^* + a^* \frac{\partial^2 \eta^*}{\partial z^2} - \frac{\sigma_{\text{el}}^* \eta^*}{\varepsilon_0^*} \frac{K_1(z^* a^*)}{K_0(z^* a^*)} - 1 \right).$$ \hspace{1cm} (27)

Obtaining the term representing the electrical stresses, which is the last term in the previous equation, constitutes an important part of the study. The details of its derivation are given in Sec. 2.2. In the previous equation, $K_0$ and $K_1$ are the modified Bessel functions of the second kind, of order zero and one, respectively. The symbol $\sigma_{\text{el}}^*$ denotes the surface electric charge density, while $\varepsilon_0^*$ is the electric constant. Surface tension between the liquid and gas is denoted by $\sigma^*$. When the previous equation is made nondimensional, the following expression for the normal-stress matching condition is obtained:

$$\hat{p}_1 = \hat{p}_2 - \frac{1}{We} \left( \eta + a^2 \frac{\partial^2 \eta}{\partial z^2} \right) - E_1 \eta \left[ \frac{K_1(z a^*)}{K_0(z a^*)} - 1 \right].$$ \hspace{1cm} (28)

In the preceding expression, $We = \rho \omega U_{\text{turb}}^2 a^\star / \sigma^*$ is the Weber number. The effect of the charged liquid jet is represented by the dimensionless group $E_1 = \sigma_{\text{el}}^* / \rho \omega U_{\text{turb}}^2 a^\star \varepsilon_0^*$. Equation (28) is an extended version of Eq. (12) by Ibrahim and Marshall [2], accounting for electrical stresses in addition to stresses due to surface tension. While the term representing the surface tension is always stabilizing, especially for short waves due to their higher curvature, the term representing the electrical stresses can be stabilizing and destabilizing, depending on whether the waves are short or long. This aspect is discussed further in Sec. 3.

The motion of the perturbed interface can be expressed as follows:

$$\eta = \eta_o e^{i(z - ct)}.$$ \hspace{1cm} (29)

When Eqs. (16), (17), and (29) are used in Eqs. (25) and (26), the perturbation velocity amplitude functions can be found for the liquid and gas layers as follows:

$$\psi_1 = (1 - b - c)\eta_o,$$ \hspace{1cm} (30)

$$\psi_2 = -c \eta_o.$$ \hspace{1cm} (31)

When these expressions are substituted into Eq. (24) for $r = 1$, the constants $c_1$ and $c_2$ are obtained as follows:

$$c_1 = \frac{1 - b - c}{I_1(z)} \eta_o,$$ \hspace{1cm} (32)

$$c_2 = -\frac{c}{K_1(z)} \eta_o.$$ \hspace{1cm} (33)

The next step involves finding expressions representing the pressure perturbations, where the Fourier modes are used as before

$$\hat{p}_j = P_j e^{i(z - ct)}.$$ \hspace{1cm} (34)

For the liquid jet, if Eqs. (16), (17), (30), (32), and (34) are used in Eq. (11), an expression for the pressure perturbation at the interface $r = 1$ is obtained as

$$\hat{p}_1 = -\frac{1 - b - c}{I_1(z)} \eta_o \left[ (1 - b - c) \eta_o - 2b I_1(z) \right] e^{i(z - ct)}.$$ \hspace{1cm} (35)

In the preceding expression, $I_0$ is the modified Bessel function of the first kind of order zero. In a similar way, when Eqs. (16), (17), (31), (33), and (34) are used in Eq. (14), an expression for the pressure perturbation at the interface $r = 1$ for the gas is achieved

$$\hat{p}_2 = -\frac{P_2}{P_1} \frac{c^2}{K_1(z)} \eta_o \eta_0 e^{i(z - ct)}.$$ \hspace{1cm} (36)
Finally, the dispersion relation is obtained after Eqs. (35) and (36) are substituted into the normal stress matching condition given in Eq. (28) and rearrangements are made

\[- (1 - b - c) \frac{\partial I_0(x)}{\partial r} - 2b(1 - b - c) \frac{\partial I_1(x)}{\partial r} - 2c(1 - b - c) x \frac{x}{K_1(x)} - \frac{\partial K_0(x)}{\partial r} - E_1 \left[ \frac{K_0(x)}{K_1(x)} - 1 \right] = 0 \quad (37)\]

The solution of the interfacial instability problem of the flow system described in Fig. 1 involves the solution of the preceding dispersion relation for the complex wave speed \(c\) as a function of \(r = \rho'_l/\rho'_a\) (the gas/liquid density ratio), \(W_e\) (the Weber number), \(E_1\) (a dimensionless parameter representing the electrical charge of the liquid jet) and \(b\) (the velocity profile parameter).

In order to solve the dispersion relation given in Eq. (37), a computer program has been written in FORTAN language. In order to solve for the complex wave speed \(c\), a method based on finding the maxima and minima of a function by using only the function values, i.e., the simplex method, has been used [13]. The Bessel functions appearing in Eq. (37) have also been computed using the same reference.

### 2.2 Derivation of the Term Representing the Electric Charge in the Normal Stress Matching Condition

The electric potential at point \(P^*\) at a distance \(r^*\) from the charged jet axis has two components:

1. The electric potential of the charged liquid jet, \(\Phi_e^*\) (mean component).
2. The perturbation in the electric potential due to the disturbed jet, \(\phi_e^*\).

Therefore, the total electrical potential at point \(P^*\) is

\[\phi_e^*(r^*, z^*, t^*) = \Phi_e^*(r^*) + \phi_e^*(r^*, z^*, t^*) \quad (38)\]

Laplace’s equation for the electric potential is given as [7]

\[\frac{\partial^2 \phi_e^*}{\partial r^2} + \frac{1}{r^*} \frac{\partial \phi_e^*}{\partial r^*} + \frac{\partial^2 \phi_e^*}{\partial z^2} = 0 \quad (39)\]

When this equation is written for the mean component of the electric potential, i.e., \(\Phi_e^*\), the term with the \(z^*\) derivative vanishes

\[\frac{\partial^2 \Phi_e^*}{\partial r^2} + \frac{1}{r^*} \frac{\partial \Phi_e^*}{\partial r^*} = 0 \quad (40)\]

The boundary condition at the surface of the undisturbed jet, i.e., \(r^* = a^*\), is as follows:

\[\Phi_e^* = \Phi_e^{*0} \quad \text{at} \quad r^* = a^* \quad (41)\]

With this condition, the solution of Eq. (40) becomes

\[\Phi_e^* = \Phi_e^{*0} \ln \frac{r^*}{a^*} \quad (42)\]

Now, Eq. (39) is written for the perturbation component \(\phi_e^*\)

\[\frac{\partial^2 \phi_e^*}{\partial r^2} + \frac{1}{r^*} \frac{\partial \phi_e^*}{\partial r^*} + \frac{\partial^2 \phi_e^*}{\partial z^2} = 0 \quad (43)\]

For the perturbation electric potential, a Fourier mode representation is employed

\[\tilde{\phi_e^*}(r^*, z^*, t^*) = \tilde{\phi_e^*}(r^*) e^{i \alpha (z^* - c^* t^*)} \quad (44)\]

In the preceding expression, \(\tilde{\phi_e^*}(r^*)\) represents the disturbance amplitude function. When Eq. (44) is used in Eq. (43), the following equation is obtained:

\[\frac{\partial^2 \tilde{\phi_e^*}}{\partial r^2} + \frac{1}{r^*} \frac{\partial \tilde{\phi_e^*}}{\partial r^*} - c^* \frac{\partial \tilde{\phi_e^*}}{\partial t^*} = 0 \quad (45)\]

The general solution of this equation is, in terms of the Bessel functions

\[\tilde{\phi_e^*} = c_1 I_0(a^* r^*) + c_2 K_0(a^* r^*) \quad (46)\]

However, \(\tilde{\phi_e^*}\) must be finite when \(r^* \to \infty\), therefore \(c_1 = 0\). When the two components of the electric potential (Eqs. (42) and (46)) are substituted into Eq. (38), the following expression is obtained:

\[\phi_e^* = \Phi_e^{*0} \ln \frac{r^*}{a^*} + c_2 K_0(a^* r^*) \quad (47)\]

The following condition is employed in order to find the constant \(c_2^*\):

\[\phi_e^* = \Phi_e^{*0} \quad \text{at} \quad r^* = a^* + \eta^* \quad (48)\]

Keeping only the terms of order \(\eta^*\) yields

\[c_2^* = - \frac{\eta^*}{a^*} \Phi_e^{*0} \ln \frac{a^*}{a^*} \quad (49)\]

When Eq. (49) is used in Eq. (47)

\[\phi_e^* = \Phi_e^{*0} \ln \frac{r^*}{a^*} + \frac{\eta^*}{a^*} K_0(a^* r^*) \quad (50)\]

Using this, the electric charge density at the jet surface is obtained as follows:

\[\sigma_e^* = - \varepsilon_e^* \frac{\partial \phi_e^*}{\partial r^*} = \varepsilon_e^* \Phi_e^{*0} \ln \frac{r^*}{a^*} + \frac{\eta^*}{a^*} K_1(a^* r^*) \quad (51)\]

The electric charge density at the surface of the undisturbed jet \(r^* = a^*\) is the first term of the preceding equation

\[\sigma_e^* = - \varepsilon_e^* \frac{\Phi_e^{*0}}{a^*} \ln \frac{r^*}{a^*} \quad (52)\]

The components of the electric field can be found by using Eq. (50)

\[E_e^* = - \frac{\partial \phi_e^*}{\partial z^*} = \frac{\Phi_e^{*0}}{a^*} \ln \frac{r^*}{a^*} + \frac{\eta^*}{a^*} K_1(a^* r^*) \quad (53)\]

\[E_{o-e}^* = - \frac{\partial \phi_e^*}{\partial z^*} = \frac{\Phi_e^{*0}}{a^*} \ln \frac{r^*}{a^*} + \frac{\eta^*}{a^*} K_0(a^* r^*) \quad (54)\]

At the disturbed liquid-jet interface, the normal component of the electrical stress is expressed as follows [7]:

\[\sigma_{e-n}^* = \frac{\varepsilon_e^*}{2} (E_{e-x}^* - E_{o-e}^*) \quad (55)\]

When Eqs. (53) and (54) are used in the preceding equation, the following expression is obtained for the perturbation normal stress:

\[\sigma_{e-n}^* = \frac{\varepsilon_e^*}{2} \eta^* \left[ \frac{\Phi_e^{*0}}{a^*} \ln \frac{r^*}{a^*} + \frac{\eta^*}{a^*} K_1(a^* r^*) \right] - 1 \quad (56)\]

Notice that the preceding equation is the last term of Eq. (27).
3 Results and Discussion

The baseline configuration for the parametric analysis corresponds to $We = 10,000$, $r = 0.01$, and $b = 0.2$. The combination of the Weber number and density ratio roughly corresponds to a liquid-gas system, such as a fuel-air injection system. These are also the same values chosen by Ibrahim and Marshall [2], allowing for a qualitative and quantitative comparison of the results.

Figure 3 shows the effect of the laminar velocity profile parameter on the amplification rate, defined as $\alpha_{ci} = \omega_k^2$, as a function of the wave number $\alpha$. It is observed that as the velocity profile becomes more curved (increasing $b$), the flow becomes stable and, furthermore, becomes unconditionally stable for $b > 0.5$. Here, smaller values of $b$ (flat profiles) correspond to higher velocity differences at the liquid-gas interface and the pressure forces working to amplify the interface become more pronounced against the surface tension forces working to stabilize it. Also, the waves corresponding to the maximum amplification become shorter as the profile becomes more flat, i.e., smaller values of $b$, implying that the resulting droplets will be smaller when the jet undergoes atomization. These inferences are in agreement with those of Ibrahim and Marshall [2] and the quantitative difference is due to the different ways velocity is scaled in Eq. (14) in the current study and that of Ibrahim and Marshall.

Figure 4 depicts the effect of the Weber number on the instability characteristics. Notice that the Weber numbers depicted correspond to the atomization regime, since $rWe \gg 1$. It can be seen that, with the increasing Weber number, i.e., decreasing stabilizing surface tension effects with respect to destabilizing inertia effects, the flow becomes more unstable. As the Weber number decreases, the unstable wavenumbers continuously migrate to longer waves, which is expected, since the surface tension is more pronounced for short waves due to the higher curvature of the latter. These two inferences imply that increasing inertia (increasing the Weber number), will result in smaller droplets due to short wave instability.

In order to contrast the role of surface tension in the atomization and Rayleigh regimes, the curves in Fig. 4 are replotted in Fig. 5 alongside three new curves for $We = 1, 10$, and $20$, the latter corresponding to $rWe < 1$, i.e., the Rayleigh regime. In Fig. 5, the long wave Rayleigh instability is evident. In the Rayleigh mode, the wavelengths that result are roughly six times the jet radius, while in the atomization regime they are 1 to 10 times as small. In contrast to the atomization regime, the role of the Weber number is reversed. In the Rayleigh regime, the surface tension destabilizes the flow and in contrast to the formation of small droplets in the atomization regime, the jet breaks up into droplets.
that are much larger, such as in the flow of ink in an ink-jet printer, due to capillary pinch-off [14]. A closer investigation of Eq. (26) provides further insight into this observation. In the atomization regime \((x \gg 1)\) the term \(-\alpha(1-x^2)/We > 0\) is in contrast to all other terms in the equation and is stabilizing. As the Weber number increases in magnitude, the stabilizing effect of this term diminishes and the flow becomes more unstable.

On the contrary, in the Rayleigh regime \((x < 1)\) the term \(-\alpha(1-x^2)/We < 0\) and is now destabilizing. As the Weber number increases, the destabilizing effect of this term diminishes and the flow becomes more stable.

In Fig. 6, the effect of the density ratio on the stability characteristics is illustrated. It can be seen that increasing the gas density (higher values of \(r\)) stabilizes the flow. It is interesting that the effect of increasing liquid density and the Weber number have the same effect.

Figure 7 illustrates the effect of an electrified jet on the atomization mode. It is clearly seen that an electrified jet destabilizes the flow and instability occurs at shorter waves as the magnitude of the electric charge density increases. Hence, the droplets that will be created upon atomization will be smaller. This result can be explained as follows. In Eq. (19), the term responsible for the electric charges appears as an additional surface tension term.

Therefore, increasing the surface charge density of the jet is, in a way, modifying the surface tension term, which explains the behavior observed in Fig. 7.

In Fig. 8, where the effect of an electrified jet on the Rayleigh mode is shown, we observe a complex and unexpected behavior. The increasing electric charge density of the jet enhances the instability as in the atomization mode. Furthermore, the lower limit of unstable wavenumbers shifts to higher values with the increasing electric charge density, extending the Rayleigh mode of instability to \(x > 1\). In other words, low wavenumbers are stabilized with increasing electric charge density. Since the electric charge term appears in the normal stress matching condition just as in the surface tension, one would expect it have the same effect as the surface tension for the Rayleigh mode as well, i.e., to stabilize the flow. However, a closer investigation of Eq. (37) and Fig. 8 provides further explanation. In the atomization regime \((x \gg 1)\), \(K_1(x)/K_0(x) > 1\) and the term \(-E_1[xK_1(x)/K_0(x)] - 1 < 0\) (destabilizing). On the contrary, in the Rayleigh regime, the term \(-E_1[xK_1(x)/K_0(x)] - 1 < 0\) (destabilizing) as \(K_1(x)/K_0(x) > 1\) for \(x > 1\). However, for \(x < 1\), although \(K_1(x)/K_0(x) > 1\), \(-E_1[xK_1(x)/K_0(x)] - 1 > 0\) (stabilizing), which explains the stabilization of low wavenumbers with increasing electrical charge density.
Fig. 7 Effect of the electrical charges on the amplification rates of the atomization mode ($r = 0.01, We = 10,000, b = 0.2$)

Fig. 8 Effect of electric charges on the amplification rates of the Rayleigh mode ($r = 0.01, We = 1, b = 0.2$)

Fig. 9 Critical values of the velocity profile parameter for the unconditional stability of the atomization mode ($r = 0.01$)
Figure 9 summarizes the variation of the critical value of the velocity profile shape parameter $b$ for unconditional stability as a function of the Weber number $We$. Comparing the curves for $E_1 = 0$ and $E_1 = 10$, it can be noticed that the critical value of $b$ increases for larger $E_1$ due to the destabilizing effect of the latter. However, the effect is more pronounced for low values of $We$, where the effect of surface tension is greater than that of inertia. This is expected because, as previously mentioned, the term representing the electrical charges appears as a modification of the surface tension term in the normal stress matching condition. At a low $We$, the stabilizing effect of the surface tension seems to dominate the stabilizing effect of the velocity profile shape and, since both the surface tension and the electrical charges are interfacial phenomena, the destabilizing effect of the electrical charges is more pronounced for low values of the Weber number.

4 Conclusions

The linear stability problem of a jet flow is investigated by performing a parametric study. It is observed that the instability occurs in two distinct modes, i.e., the Rayleigh and atomization modes. The results show that while the surface tension stabilizes the atomization regime, an electrically charged jet is a destabilizing effect. On the contrary, surface tension destabilizes the Rayleigh mode, while the electrical charge is still a destabilizing effect. The effect of electrical charges on the Rayleigh mode of instability has not been numerically studied or reported elsewhere, to the author’s best knowledge.

The effect of the laminar velocity profile shape is destabilizing for flat profiles or for flows that are not fully developed. In a liquid-gas system, increasing the density of the liquid also has a destabilizing effect.

Since the sizes of the resulting droplets is strongly dependent on the lengths of the preceding instability waves, the desired droplet size can be obtained by choosing the right combination of the investigated parameters. Likewise, adjusting one parameter may enhance the instability in an otherwise stable flow and vice versa.

The analysis does not reveal the effect of jet viscosity on the stability characteristics, which obviously should have an effect. As a further study, a more comprehensive study taking the jet viscosity into account is envisioned, which should provide further insight into the problem.

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References