Stability of Boundary-Layers in Compressible Flow

AE 549 Linear Stability Theory and Laminar-Turbulent Transition
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Compressible-flow stability

• Laminar-turbulent transition is an important phenomenon in fluid dynamics and aerodynamics with a large number of engineering applications.

• The reason is that it has a very important effect on heat transfer and skin friction drag.

• The reduction of heating rates for the orbital reentry vehicles and ICBMs, the reduction of drag on the high subsonic-speed commercial aircraft wings are only few of areas that a good knowledge about transition is essential.
Compressible-flow stability

- Compressibility makes this problem not only more realistic for most flow regimes, but also fundamentally more complex.
- The studies on this aspect start with Küchemann (1938), with an effort to try to build a compressible linear stability theory.
- Lees and Lin (1946) followed with theoretical investigations.
- It has been shown that a necessary condition for the existence of an unstable disturbance is:

\[
\left[ \frac{d}{dy} \left( \rho \frac{dU}{dy} \right) \right]_{y=y_s} = 0,
\]

provided that \( U(y_s) > U_\infty - a_\infty \), where \( \rho \) is the density, \( U \) the streamwise mean velocity component of the flow, \( y \) the normal distance from the wall and \( y_s \) is the location where the above equality is satisfied. This is the **generalized inflection point theorem** and is the extension of the well-known inflection point theorem (or Rayleigh’s Theorem) in incompressible flow.
Compressible-flow stability

• Mack outlined a complete numerical investigation for compressible laminar boundary-layers and discovered higher modes at supersonic speeds. Whenever the following condition holds:

\[ \frac{U - cr}{a} > 1, \]

i.e. whenever there is a relative supersonic region in the flow, there exists an infinite number of unstable modes (or wave numbers).

• The first of these modes is related to the Tollmien–Schlichting mode in incompressible flow, but the higher or additional modes have no incompressible counterparts.

• The first of these additional modes has been referred to as the Mack mode.
Compressible-flow stability

- Mack outlined a complete numerical investigation for compressible laminar boundary-layers and discovered higher modes at supersonic speeds. Whenever the following condition holds:

$$\tilde{M} = \frac{U - c_r}{a} > 1,$$

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Compressible-flow stability

• Unlike incompressible flow, where a two-dimensional disturbance is the most unstable at any Reynolds number according to Squire’s theorem, for supersonic flow, the most unstable disturbance is always oblique.

• While the most unstable disturbances are oblique for the Tollmien-Schlichting mode of instability, the most unstable disturbances are two-dimensional for the Mack mode.
Compressible flow stability theory

• The mathematical model of the compressible stability problem starts with the 3-D Navier-Stokes equations for a compressible boundary-layer over an adiabatic flat plate.

• The momentum, energy, continuity equations and the equation of state for a viscous, heat conducting, perfect gas in Cartesian coordinates are subject to method of small disturbances for linearization.

• Accordingly, each instantaneous flow property, i.e. velocity, pressure, temperature, density, viscosity and thermal conductivity is split into a steady mean (basic) and an unsteady fluctuating component:

\[ \phi^*(x^*, y^*, z^*, t^*) = \phi^*(x^*, y^*, z^*) + \hat{\phi}^*(x^*, y^*, z^*, t^*) \]

where \( \phi^* \) represents any one of \( u^*, v^*, w^* \) (velocity components in Cartesian coordinates); \( p^*, T^*, \rho^* \) (pressure, temperature and density); \( \mu^* \) (viscosity) and \( k^* \) (thermal conductivity).
Compressible flow stability theory

The following assumptions are made:

- Disturbances are small so quadratic or higher order terms involving perturbation quantities are neglected,
- Parallel flow assumption, i.e. $U^* = U^*(y^*)$ and $W^* = W^*(y^*)$ only and $V^* = 0$ for the mean, basic flow.
- Temperature, $T^*$ is a function of normal distance $y^*$ only and fluid properties, $\mu^*, k^*, C_p^*, C_v^*$ are functions of temperature only.

- Velocities are non-dimensionalized by $U_e^*$ (boundary-layer edge velocity) and lengths are made dimensionless by $L^* = \sqrt{\nu^*x^* / U_e^*}$ (Blasius length scale).
- Temperature, density, pressure, viscosity and heat conduction coefficient are non-dimensionalized by their respective freestream values, $T_e^*, \rho_e^*, p_e^*, \mu_e^*, k_e^*$.
- The Reynolds number is defined as $Re = \rho_e^* U_e^* L^* / \mu_e^*$.
Compressible flow stability theory

• The mean laminar flow is assumed to be influenced by a disturbance composed of a number of normal modes, which are propagating (traveling) waves of the form:

$$\phi(x, y, z, t) = \bar{\phi}(y)e^{i(\alpha x + \beta z - \omega t)}$$

• $\alpha$ and $\beta$ are x and z components of the wave number vector $\vec{k}$,

• $\omega$ is the complex frequency defined as $\omega = \alpha \cdot c$ with $c = c_r + ic_i$ representing the complex wave velocity according to the temporal amplification formulation.

• The magnitude of the wave number vector is $k = \sqrt{\alpha^2 + \beta^2}$ and the wave angle is $\psi = \tan^{-1}(\beta/\alpha)$.

• The disturbance amplitude of the relevant variable is defined by $\bar{\phi}(y)$.

• The real part of the complex frequency $\omega_r = \alpha c_r$, is the circular frequency of the disturbance, The imaginary part of the complex frequency $\omega_i = \alpha c_i$, is the amplification rate.

• The imaginary part of the complex wave velocity $c_i$, is the amplification factor determining a stable ($c_i < 0$), a neutrally stable ($c_i = 0$), or an unstable ($c_i > 0$) disturbance, while its real part $c_r$, is the phase velocity.
Compressible flow stability theory

• Substitution of the normal modes into the dimensionless, linearized system of equations and performing necessary algebra leads to the set of perturbation equations.

• The resulting equations are then transformed into a system of first order differential equations through the following variable definitions:

\[
\begin{align*}
Z_1 &= \alpha \ddot{u} + \beta \ddot{w} & Z_2 &= Z'_1 \\
Z_3 &= \ddot{v} & Z_4 &= \ddot{p} / \gamma M^2 \\
Z_5 &= \ddot{T} & Z_6 &= Z'_5 & Z_7 &= \alpha \ddot{w} - \beta \ddot{u} & Z_8 &= Z'_7
\end{align*}
\]

where the Mach number is defined as \( M = \frac{U_e^*}{\sqrt{\gamma R^* T_e^*}} \), \( \gamma \) being the ratio of specific heats and \( R^* \) being the gas constant.

• The system of equations can be expressed as:

\[
Z'_i = \sum_{j=1}^{8} a_{ij} Z_j, \quad i = 1,8
\]

where, \( a_{ij} \) are the elements of the coefficient matrix and prime (') denotes derivate with respect to \( y \). The boundary conditions are:

\[
\begin{align*}
Z_1(0) &= Z_3(0) = Z_5(0) = Z_7(0) = 0 \quad (no \ slip), \\
Z_1, Z_3, Z_5, Z_7 &\to 0 \quad as \quad y \to \infty \quad (freestream).
\end{align*}
\]
Compressible flow stability theory

• The two-dimensional basic flow equations are solved for velocity \((U)\) and temperature \((T)\) and their derivatives.

For the velocity field:

\[
2(\mu'U' + \mu U'') + FU' = 0,
\]

For the temperature field:

\[
2\left(\frac{\mu}{Pr} T'\right)' + FT' = -2(\gamma - 1)M^2\mu(U')^2.
\]

with Prandtl number defined as \(Pr = \mu^*C_p^*/k^*\). Also notice that \(\rho T = 1\) in the current formulation.

• The boundary conditions are:

\[
F(0) = F'(0) = 0, \quad T'(0) = 0,
\]

\[
F' \to 1, T \to 1 \quad \text{as} \quad y \to \infty.
\]

• Temperature dependent fluid properties are calculated using empirical formulae for example Sutherland’s viscosity law.
Compressible flow stability theory

- Equations and the boundary conditions given constitute a characteristic value problem for the variables \((\alpha, \beta, \omega, Re)\). The problem is solved with the \textbf{Shooting Method and Gram-Schmidt Orthonormalization}.
Velocity and Temperature profiles
Generalized inflection point
Variation of the stability curves with wave orientation, $M = 4$.

\[ \psi = 0^\circ \quad \text{and} \quad \psi = 20^\circ \]
Variation of the stability curves with wave orientation for $M = 4$. 

$\psi = 60^\circ$  

$\psi = 80^\circ$
Stability curves for the most unstable wave directions, Tollmien-Schlichting mode

$M = 0, \psi = 0^\circ$

$M = 1, \psi = 0^\circ$
Stability curves for the most unstable wave directions, Tollmien-Schlichting mode

\[ M = 2, \psi = 45^\circ \quad M = 3, \psi = 55^\circ \]
Stability curves for the most unstable wave directions, Tollmien-Schlichting mode

\[ M = 4, \psi = 60^\circ \]

\[ M = 5, \psi = 60^\circ \]
Stability curves for the most unstable wave directions, Tollmien-Schlichting mode

\( M = 6, \psi = 60^\circ \)  \hspace{2cm}  \( M = 8, \psi = 60^\circ \)
Variation of critical Reynolds number and the most unstable wave angle with Mach number
Comparison of neutral stability curves for two-dimensional disturbances with literature data

\[ M = 2, \psi = 0° \]

\[ M = 4, \psi = 0° \]
Conclusions

• The results confirm that as soon as there is relative supersonic flow, a second mode of instability (Mack mode) is observed in addition to the usual Tollmien-Schlichting mode.

• The Mack mode is rapidly stabilized as wave angle is increased and only the Tollmien-Schlichting mode is observed for higher wave angles. The most unstable wave directions are typically around $\Psi = 60^\circ$ for moderate and high Mach numbers.

• The most interesting and important result is the behavior of the stability curves for $M > 4$ for oblique waves. The stability characteristics become almost independent of Mach number for $M > 4$ for three-dimensional waves (oblique waves), which are also shown to be the most unstable waves for these Mach numbers.