Reduction of Variables – Dimensional Analysis
Dimensions and Units

• Review
  – **Dimension:** Measure of a physical quantity, e.g., length, time, mass
  – **Units:** Assignment of a number to a dimension, e.g., (m), (sec), (kg)
  – **7 Primary Dimensions:**
    1. Mass \( m \) (kg)
    2. Length \( L \) (m)
    3. Time \( t \) (sec)
    4. Temperature \( T \) (K)
    5. Current \( I \) (A)
    6. Amount of Light \( C \) (candela)
    7. Amount of matter \( N \) (mol)
Dimensions and Units

• Review, continued
  – All non-primary dimensions can be formed by a combination of the 7 primary dimensions
  – Examples
    • \{Velocity\} = \{Length/Time\} = \{L/t\}
    • \{Force\} = \{Mass Length/Time\} = \{mL/t^2\}
Dimensional Homogeneity

• Law of dimensional homogeneity (DH): every additive term in an equation must have the same dimensions

• Example: Bernoulli equation

\[ p + \frac{1}{2} \rho V^2 + \rho g z = C \]

- \( \{p\} = \{\text{force/area}\} = \{\text{mass x length/time}^2 \times 1/\text{length}^2\} = \{\text{m/(t}^2\text{L)}\} \)
- \( \{1/2\rho V^2\} = \{\text{mass/length}^3 \times (\text{length/time})^2\} = \{\text{m/(t}^2\text{L)}\} \)
- \( \{\rho g z\} = \{\text{mass/length}^3 \times \text{length/time}^2 \times \text{length}\} = \{\text{m/(t}^2\text{L)}\} \)
Nondimensionalization of Equations

• Given the law of dimensional homogeneity, if we divide each term in the equation by a collection of variables and constants that have the same dimensions, the equation is rendered **nondimensional**

• In the process of nondimensionalizing an equation, nondimensional parameters often appear, e.g., Reynolds number and Froude number
Nondimensionalization of Equations

• To nondimensionalize, for example, the Bernoulli equation, the first step is to list primary dimensions of all dimensional variables and constants

\[
p + \frac{1}{2} \rho V^2 + \rho g z = C
\]

\[
\{p\} = \{m/(t^2L)\} \quad \{\rho\} = \{m/L^3\} \quad \{V\} = \{L/t\}
\]
\[
\{g\} = \{L/t^2\} \quad \{z\} = \{L\}
\]

– Next, we need to select Scaling Parameters. For this example, select \( L, U_0, \rho_0 \).
Nondimensionalization of Equations

• By inspection, nondimensionalize all variables with scaling parameters

\[ p^* = \frac{p}{\rho_0 U_0^2} \quad \rho^* = \frac{\rho}{\rho_0} \quad V^* = \frac{V}{U_0} \]
\[ g^* = \frac{gL}{U_0^2} \quad z^* = \frac{z}{L} \]

• Back-substitute \( p, \rho, V, g, z \) into dimensional equation

\[ \rho_0 U_0^2 p^* + \frac{1}{2} \rho_0 \rho^* \left( U_0^2 V^* \right) + \rho_0 \rho^* g^* U_0^2 z^* = C \]
Nondimensionalization of Equations

• Divide by $\rho_0 U_0^2$ and set $\rho^* = 1$ (incompressible flow)

\[
p^* + \frac{1}{2} V^*^2 + g^* z^* = \frac{C}{\rho_0 U_0^2} = C^*
\]

• Since $g^* = 1/Fr^2$, where $Fr = \frac{U_0}{\sqrt{gL}}$

\[
p^* + \frac{1}{2} V^*^2 + \frac{1}{Fr^2} z^* = C^*
\]
Nondimensionalization of Equations

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$$p^* + \frac{1}{2} V^*^2 + g^* z^* = \frac{C}{\rho_0 U_0^2} = C^*$$

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$$p^* + \frac{1}{2} V^*^2 + \frac{1}{Fr^2} z^* = C^*$$
Nondimensionalization of Equations

• Advantages of nondimensionalization
  – Increases insight about key parameters
  – Decreases number of parameters in the problem
    • Easier communication
    • Fewer experiments
    • Fewer simulations
  – Extrapolation of results to untested conditions
Dimensional Analysis and Similarity

• Nondimensionalization of an equation is useful only when the equation is known!

• In many real-world flows, the equations are either unknown or too difficult to solve.
  – *Experimentation* is the only method of obtaining reliable information
  – In most experiments, geometrically-scaled models are used (time and money).
  – Experimental conditions and results must be properly scaled so that results are meaningful for the full-scale prototype.
  – *Dimensional Analysis*
Dimensional Analysis and Similarity

• Primary purposes of dimensional analysis
  – To generate nondimensional parameters that help in the design of experiments (physical and/or numerical) and in reporting of results
  – To obtain scaling laws so that prototype performance can be predicted from model performance.
  – To predict trends in the relationship between parameters.
Dimensional Analysis and Similarity

- **Geometric Similarity** - the model must be the same shape as the prototype. Each dimension must be scaled by the same factor.

- **Kinematic Similarity** - velocity as any point in the model must be proportional

- **Dynamic Similarity** - *all forces* in the model flow scale by a constant factor to corresponding forces in the prototype flow.

- **Complete Similarity** is achieved only if all 3 conditions are met. This is not always possible.
Complete similitude can be achieved by enforcing geometric similitude and making the dimensionless groupings of the model the same as the desired prototype.
Buckingham $\Pi$ Theorem

• The number of dimensionless groups of variables needed to correlate is $j$ less than the number of dimensional quantities ($n$), where $j$ is the number of independent dimensions
Dimensional Analysis and Similarity

- Complete similarity is ensured if all independent $\Pi$ groups are the same between model and prototype.
- What is $\Pi$? (Buckingham’s pi Theorem)
  - We let uppercase Greek letter $\Pi$ denote a nondimensional parameter, e.g., Reynolds number $Re$, Froude number $Fr$, Drag coefficient, $C_D$, etc.
- Consider automobile experiment
- Drag force is $F = f(V, \rho, \mu, L)$
- Through dimensional analysis, we can reduce the problem to
  $$\Pi_1 = f(\Pi_2) \rightarrow C_D = f(Re)$$
Method of Repeating Variables

- Non-dimensional parameters $\Pi$ can be generated by several methods.
- We will use the Method of Repeating Variables.
- Six steps
  1. List the parameters in the problem and count their total number $n$.
  2. List the primary dimensions of each of the $n$ parameters.
  3. Set the reduction $j$ as the number of primary dimensions. Calculate $k$, the expected number of $\Pi$'s, $k = n - j$.
  4. Choose $j$ repeating parameters.
  5. Construct the $k$ $\Pi$'s, and manipulate as necessary.
  6. Write the final functional relationship and check algebra.
Example

Ball Falling in a Vacuum

- Step 1: List relevant parameters. $z = f(t, w_0, z_0, g) \Rightarrow n = 5$
- Step 2: Primary dimensions of each parameter:
  
  $\begin{align*}
  z & \in \{L^1\} \\
  t & \in \{t^1\} \\
  w_0 & \in \{L^1t^{-1}\} \\
  z_0 & \in \{L^1\} \\
  g & \in \{L^1t^{-2}\}
  \end{align*}$
- Step 3: As a first guess, reduction $j$ is set to 2 which is the number of primary dimensions ($L$ and $t$). Number of expected $\Pi$'s is $k = n - j = 5 - 2 = 3$
- Step 4: Choose repeating variables $w_0$ and $z_0$
Guidelines for choosing Repeating parameters

1. Never pick the dependent variable. Otherwise, it may appear in all the Π's.
2. Chosen repeating parameters must not by themselves be able to form a dimensionless group. Otherwise, it would be impossible to generate the rest of the Π's.
3. Chosen repeating parameters must represent all the primary dimensions.
4. Never pick parameters that are already dimensionless.
5. Never pick two parameters with the same dimensions or with dimensions that differ by only an exponent.
6. Choose dimensional constants over dimensional variables so that only one Π contains the dimensional variable.
7. Pick common parameters since they may appear in each of the Π's.
8. Pick simple parameters over complex parameters.
Example, continued

- Step 5: Combine repeating parameters into products with each of the remaining parameters, one at a time, to create the $\Pi$’s.
  - $\Pi_1 = zw_0^{a1}z_0^{b1}$
    - $a1$ and $b1$ are constant exponents which must be determined.
    - Use the primary dimensions identified in Step 2 and solve for $a1$ and $b1$.
  
  \[
  \{\Pi_1\} = \{L^0t^0\} = \{zw_0^{a1}z_0^{b1}\} = \{L^1(L^1t^{-1})a_1L^{b1}\}
  \]
  - Time equation: $\{t^0\} = \{t^{-a1}\} \rightarrow 0 = -a_1 \rightarrow a_1 = 0$
  - Length equation:
    \[
    \{L^0\} = \{L^1L^{a1}L^{b1}\} \rightarrow 0 = 1 + a_1 + b_1 \rightarrow b_1 = -1 - a_1 \rightarrow b_1 = -1
    \]
  - This results in
    \[
    \Pi_1 = zw_0^{0}z_0^{-1} = \frac{z}{z_0}
    \]
Example, continued

- Step 5: continued
  - Repeat process for $\Pi_2$ by combining repeating parameters with $t$
  - $\Pi_2 = tw_0^{a_2}z_0^{b_2}$
    
    $$\{\Pi_2\} = \{L^0t^0\} = \{tw_0^{a_2}z_0^{b_2}\} = \{t^1(L^1t^{-1})^{a_2}L^{b_2}\}$$

  - Time equation:
    $$\{t^0\} = \{t^1t^{-a_2}\} \rightarrow 0 = 1 - a_2 \rightarrow a_2 = 1$$

  - Length equation:
    $$\{L^0\} = \{L^{a_2}L^{b_2}\} \rightarrow 0 = a_2 + b_2 \rightarrow b_2 = -a_2 \rightarrow b_2 = -1$$

  - This results in
    $$\Pi_2 = tw_0^{1}z_0^{-1} = \frac{w_0t}{z_0}$$
Example, continued

- Step 5: continued
  - Repeat process for $\Pi_3$ by combining repeating parameters with $g$
    - $\Pi_3 = gw_0^{a_3}z_0^{b_3}$
      \[\{\Pi_3\} = \{L^0 t^0\} = \{gw_0^{a_3} z_0^{b_3}\} = \{L^1 t^{-2}(L^1 t^{-1})^{a_3} L^{b_3}\}\]
  - Time equation:
    \[\{t^0\} = \{t^{-2}t^{-a_3}\} \rightarrow 0 = -2 - a_3 \rightarrow a_3 = -2\]
  - Length equation:
    \[\{L^0\} = \{L^1 L^{a_3} L^{b_3}\} \rightarrow 0 = 1 + a_3 + b_3 \rightarrow b_3 = -1 - a_3 \rightarrow b_3 = 1\]
  - This results in
    \[\Pi_3 = gw_0^{-2} z_0^{1} = \frac{g z_0}{w_0^2}\]
    \[\Pi_{3, \text{modified}} = \left(\frac{g z_0}{w_0^2}\right)^{-1/2} = \frac{w_0}{\sqrt{g z_0}} = Fr\]
Example, continued

• Step 6:
  – Double check that the $\Pi$'s are dimensionless.
  – Write the functional relationship between $\Pi$'s

$$\Pi_1 = f(\Pi_2, \Pi_3) \rightarrow \frac{z}{z_0} = f\left(\frac{w_0 t}{z_0}, \frac{w_0}{\sqrt{g z_0}}\right)$$

  – Or, in terms of nondimensional variables

$$z^* = f(t^*, Fr)$$

• Overall conclusion: Method of repeating variables properly predicts the functional relationship between dimensionless groups.

• However, the method cannot predict the exact mathematical form of the equation.
Experimental Testing and Incomplete Similarity

- One of the most useful applications of dimensional analysis is in designing physical and/or numerical experiments, and in reporting the results.
- Setup of an experiment and correlation of data.

Consider a problem with 5 parameters: one dependent and 4 independent.

- Full test matrix with 5 data points for each independent parameter would require $5^4=625$ experiments!!
- If we can reduce to 2 $\Pi$'s, the number of independent parameters is reduced from 4 to 1, which results in $5^1=5$ experiments vs. 625!!
Finding Dimensionless Groupings-1

- Consider viscous flow past a sphere

![Diagram of viscous flow past a sphere](image)

- There are 5 dimensional quantities

  - $F$ = force on the sphere ($MLT^{-2}$)
  - $\mu$ = viscosity of the fluid ($ML^{-1}T^{-1}$)
  - $\rho$ = density of the fluid ($ML^{-3}$)
  - $U$ = velocity of the fluid ($LT^{-1}$)
  - $D$ = diameter of the sphere ($L$)

- There are 3 independent dimensions, i.e., $M$, $L$, and $T$

- The Buckingham $\pi$ Theorem tells us there are $5 - 3 = 2$ dimensionless groupings