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1 Introduction

In 1807, the French mathematician Joseph Fourier (1768-1830) submitted a paper to the Academy of Sciences in Paris. In it he presented a mathematical description of problems involving heat conduction. Although the paper was at first rejected, it contained ideas that would develop into an important area of mathematics named in honor, Fourier analysis. One surprising ramification of Fourier's work was that many familiar functions can be expanded in infinite series and integrals involving trigonometric functions. The idea today is important in modeling many phenomena in physics and engineering.

2 Fourier Series

Definition 1 (Periodic functions)
A function $f(t)$ ist said to have a period *T or to be* periodic *with period T if for all t, $f(t+T) = f(t)$, where T is a positive constant. The least value of $T > 0$ is called the* principal period *or the* fundamental period *or simply the* period *of $f(t)$.*

Example 1
The function $\sin x$ has periods $2\pi, 4\pi, 6\pi, \ldots$, since $\sin(x+2\pi), \sin(x+4\pi), \sin(x+6\pi), \ldots$ all equal $\sin x$.

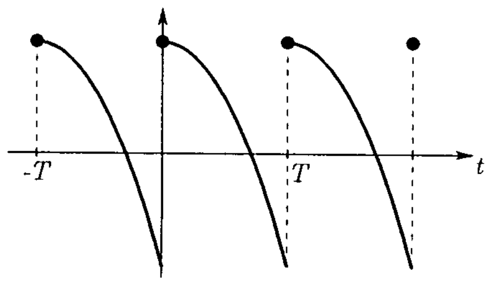
Example 2
Let $a \in \mathbb{R}$. If f(x) has the period 2π then $F(t) := f(\omega t) := f(\frac{2\pi}{T}t)$ has the period T. (substitute $\frac{2\pi}{T}t := x, \quad \omega := \frac{2\pi}{T}$)

Example 3
If f has the period T then

$$\int_a^{a+T} f(t)dt = \int_0^T f(t)dt \quad \forall a \in \mathbb{R}$$

Definition 2 (Periodic expansion)
Let a function f be declared on the interval $[0,T)$. The periodic expansion *$\tilde{f}$ of f is defined by the formula*

$$\tilde{f}(t) = \begin{cases} f(t) & 0 \leq t < T \\ \tilde{f}(t-T) & \forall t \in \mathbb{R} \end{cases}$$



Definition 3 (Piecewise continuous functions)
A function f defined on $I = [a, b]$ is said to be <u>*piecewise continuous*</u> *on I if and only if*

(i) there is a subdivision $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ such that f is continuous on each subinterval $I_k = \{x : x_{k-1} < x < x_k\}$ and

(ii) at each of the subdivision points $x_0, x_1, \ldots, x_n$ both one-sided limits of f exist.

Theorem 1
Let f be continuous on $I = [-\pi, \pi]$. Suppose that the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\ nx + b_n \sin nx) \tag{1}$$

converges uniformly to f for all $x \in I$. Then

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt \quad n = 0, 1, 2, \ldots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ntdt \quad n = 1, 2, \ldots \end{aligned} \tag{2}$$

Proof. The partial sums s_k are defined as

$$s_k(x) = \frac{a_0}{2} + \sum_{m=1}^{k}(a_m \cos\ mx + b_m \sin mx)$$

Since the sequence $s_k(x)$ converges uniformly to $f(x)$, it follows that $s_k(x)\cos nx$ converges uniformly to $f(x)\cos nx$ as $k \to \infty$ for each fixed n.
(Observe that $|s_k(x)\cos nx - f(x)\cos nx| \leq |s_k(x) - f(x)|$)

Therefore, for each fixed n

$$f(x)\cos nx = \frac{a_0}{2}\cos nx + \sum_{m=1}^{\infty}(a_m \cos\ mx\cos nx + b_m \sin mx \cos nx)$$

The uniformly convergent series may be integrated term-by-term

$$\int_{\pi}^{\pi} f(x)\cos nx dx = \pi a_n, \qquad n = 0, 1, 2, \ldots$$

The argument goes analog for $f(x)\sin nx$. $\square$

Definition 4 (Fourier coefficients, Fourier series)
The numbers a_n and b_n are called the <u>*Fourier coefficients*</u> *of f. When a_n and b_n are given by (2), the trigonometric series (1) is called the* <u>*Fourier series*</u> *of the function f.*

Remark 1
If f is any integrable function then the coefficients a_n and b_n may be computed. However, there is no assurance that the Fourier series will converge to f if f is an arbitrary integrable function. In general, we write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\ nx + b_n \sin nx)$$

to indicate that the series on the right may or may not converge to f at some points.

Remark 2 (Complex Notation for Fourier series)
Using Euler's identities,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where i is the imaginary unit such that $i^2 = -1$, the Fourier series of $f(x)$ can be written in complex form as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{3}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \tag{4}$$

and

$$c_0 = \frac{1}{2}a_0, \quad c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \frac{1}{2}(a_n + ib_n), \qquad n = 1, 2, \ldots$$

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}), \qquad n = 1, 2, \ldots$$

Example 4
Let $f(x)$ be defined in the interval $[0, T]$ and determined outside of this interval by its periodic extension, i.e. assume that $f(x)$ has the period T. The Fourier series corresponding to $f(x)$ (with $\omega := \frac{2\pi}{T}$) is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega x + b_n \sin n\omega x) \tag{5}$$

where the Fourier coefficients a_n and b_n are

$$a_n = \frac{2}{T}\int_0^T f(x)\cos n\omega x dx \qquad n = 0, 1, 2, \ldots \tag{6}$$

$$b_n = \frac{2}{T}\int_0^T f(x)\sin n\omega x dx \qquad n = 1, 2, \ldots \tag{7}$$

Example 5
Let a_n and b_n be the Fourier coefficients of f. The <u>phase angle form</u> of the Fourier series of f is

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega x + \delta_n)$$

with

$$c_n = \sqrt{a_n^2 + b_n^2} \qquad n = 1, 2, \ldots$$

and

$$\delta_n = \tan^{-1}(-\frac{b_n}{a_n}), \qquad n = 1, 2, \ldots$$

Example 6
We compute the Fourier series of the function f given by

$$f(x) = \begin{cases} 1 & , 0 \le x < \pi \\ -1 & , \pi \le x < 2\pi \end{cases}$$

Since f is an odd function, so is $f(x)\cos nx$, and therefore

$$a_n = 0, \qquad n = 1, 2, 3, \ldots$$

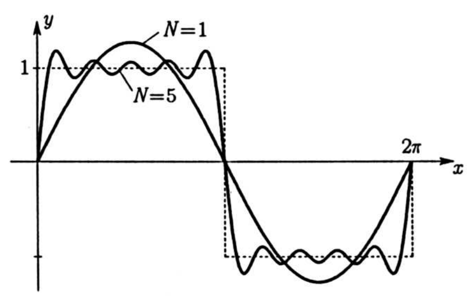
$$a_0 = 0$$

For $n \ge 1$ the coefficient b_n is given by

$$b_n = \frac{1}{\pi}\left(\int_0^{\pi} \sin nx dx - \int_{\pi}^{2\pi} \sin nx\right) = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

It follows

$$f \sim \frac{4}{\pi}(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots)$$



2.1 Dirichlet conditions

It is important to establish simple criteria which determine when a Fourier series converges. In this section we will develop conditions on $f(x)$ that enable us to determine the sum of the Fourier series. One quite useful method to analyse the convergence properties is to express the partial sums of a Fourier series as integrals. Riemann and Fejer have since provided other ways of summing Fourier series. In this section we limit the study of convergence to functions that are piecewise smooth on a given interval.

Definition 5 (Piecewise smooth function)
A function f is piecewise smooth on an interval if both f and f' are piecewise continuous on the interval.

Theorem 2
Suppose that f is piecewise smooth and periodic.
Then the series (1) with coefficients (2) converges to

1. *$f(x)$ if x is a point of continuity.*
2. *$\frac{1}{2}(f(x+0)+f(x-0))$ if x is a point of discontinuity.*

This means that, at each x between $-L$ and L, the Fourier series converges to the average of the left and the right limits of $f(x)$ at x. If f is continuous at x, then the left and the right limits are both equal to $f(x)$, and the Fourier series converges to $f(x)$ itself. If f has a jump discontinuity at x then the Fourier series converges to the point midway in the gap at this point.

Remark 3
Let f be a given piecewise continuous function. We say that f is standardised if its values at points x_i of discontinuity are given by

$$f(x_i) = \frac{1}{2}[f(x_i+) + f(x_i-)]$$

Remark 4
The conditions imposed on $f(x)$ are sufficient but not necessary, i.e if the conditions are satisfied the convergence is guaranteed. However, if they are not satisfied the series may or may not converge.

Theorem 3 (Bessel's inequality)
Suppose that f is integrable on the interval $[0,T]$. Let a_n, b_n, c_n be the Fourier coefficients of f. Then

$$\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) = 2\sum_{k=-\infty}^{\infty}|c_k|^2 \leq \frac{2}{T}\int_0^T |f(t)|^2 dt \qquad (8)$$

Proof. S_N denotes the partial sums $S_N = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n \cos n\omega x + b_n \sin n\omega x)$ Now

$$\int [f(t) - S_N(t)]^2 dt = \int f^2(t)dt - 2\int f(t)S_N(t)dt + \int S_N^2(t)dt$$

From the definition of the Fourier coefficients, it follows that

$$\frac{1}{2}a_0^2 + \sum_{k=1}^{N}(a_k^2 + b_k^2) = \frac{2}{T}\int_0^T f(t)S_N(t)dt$$

Also, by multiplying out the terms of $S_N^2(t)$ and taking into account the orthogonality relations of the trigonometric functions, it follows that

$$\frac{2}{T}\int_0^T S_N^2(t)dt = \frac{2}{T}\int_0^T f(t)S_N(t)dt$$

Therefore

$$0 \leq \frac{2}{T}\int_0^T [f(t) - S_N(t)]^2 dt = \frac{2}{T}\int_0^T f^2(t)dt - \left\{\frac{1}{2}a_0^2 + \sum_{k=1}^{N}(a_k^2 + b_k^2)\right\}$$

Since f^2 is integrable we may let N tend to infinity

$$\frac{1}{2}a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \leq \frac{2}{T}\int_0^T f^2(t)dt < \infty$$

□

Theorem 4 (Riemann lemma)
Let f be integrable and a_n and b_n be the Fourier coefficients of f. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$$

which means

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\cos nt dt = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\sin nt dt = 0$$

Proof. From the Bessel inequality it follows that

$$\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) < \infty$$

and therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$$

□

Theorem 5 (Parseval's identity)

$$\frac{2}{T}\int_0^T |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{9}$$

if a_n and b_n are the Fourier coefficients corresponding to $f(x)$ and if $f(x)$ satisfies the Dirichlet conditions.

2.2 The Gibbs Phenomenon

Near a point, where f has a jump discontinuity, the partial sums S_n of a Fourier series exhibit a substantial overshoot near these endpoints, and an increase in n will not diminish the amplitude of the overshoot, although with increasing n the the overshoot occurs over smaller and smaller intervals. This phenomenon is called Gibbs phenomenon. In this section we examine some detail in the behaviour of the partial sums S_n of $S(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k}$.

Theorem 6

$$\sum_{k=1}^{\infty} \frac{\sin kx}{x} = \frac{\pi - x}{2}, \quad 0 < x < 2\pi$$

Proof. For $0 < x < 2\pi$ and for each $n \in \mathbb{N}$

$$\frac{1}{2} + \sum_{k=1}^{n} \cos(kx) = \frac{\sin(\frac{2n+1}{2}x)}{2\sin(\frac{x}{2})}, \qquad 0 < x < 2\pi$$

Therefore

$$\begin{aligned} S_n = \sum_{k=1}^{n} \frac{\sin kx}{k} &= \int_\pi^x \sum_{k=1}^{n} \cos(kt)dt \\ &= \int_\pi^x \left(\frac{-1}{2} + \frac{\sin(\frac{2n+1}{2}t)}{2\sin(t/2)} \right) dt \\ &= \frac{\pi - x}{2} + \frac{1}{2n+1} \left(-\frac{\cos(\frac{2n+1}{2})x}{\sin\frac{x}{2}} \Big|_\pi^x - \frac{1}{2}\int_\pi^x \frac{\cos(t/2)}{\sin^2(t/2)} \cos(\frac{2n+1}{2}t)dt \right) \end{aligned}$$

Since

$$\left|\int_{\pi}^{x} \frac{\cos(t/2)}{\sin^2(t/2)}\right| dt = \frac{2}{\sin(x/2)} - 2$$

we have

$$\left|\sum_{k=1}^{n} \frac{\sin kx}{k} - \frac{\pi - x}{2}\right| \leq \frac{1}{2n+1}\left(\frac{2}{\sin\frac{x}{2}} - 1\right) \to 0 \qquad \text{if } n \to \infty$$

$\square$

The next step is to replace the partial sums S_n with integrals

$$S_n(x) = \int_0^x \sum_{k=1}^{n} \cos(kt)dt = \frac{-x}{2} + \int_0^x \frac{\sin(\frac{2n+1}{2}t)}{2\sin(t/2)}dt \to \frac{\pi - x}{2} \qquad (n \to \infty)$$

For $x \approx 0$ we have a typically "overshoot". This will be the next step to show. Let $x_n = \frac{2\pi}{2n+1}$.

$$S_n(x_n) + \frac{1}{2}x_n = \int_0^{x_n} \frac{\sin(\frac{2n+1}{2}t)}{2\sin(t/2)}dt = \int_0^{\pi} \frac{\sin\tau}{\sin(\frac{\tau}{2n+1})(2n+1)}d\tau \to \int_0^{\pi} \frac{\sin\tau}{\tau}d\tau \qquad (n \to \infty)$$

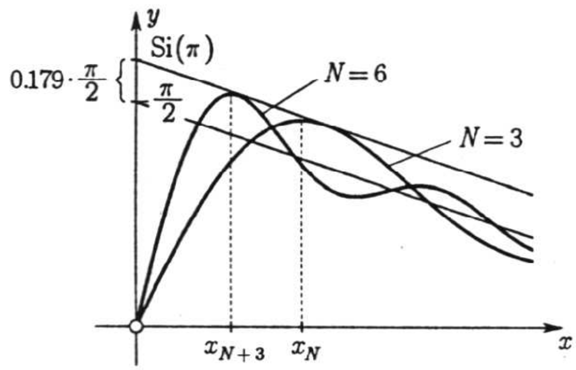
Theorem 7 (The Gibbs phenomenon)
Let $n \in \mathbb{N}$ and $x_n = \frac{2\pi}{2n+1}$.

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{\sin(kx_n)}{k}\right) = \int_0^{\pi} \frac{\sin\tau}{\tau}d\tau$$

and

$$\int_0^{\pi} \frac{sin\tau}{\tau}d\tau = \frac{\pi}{2} \cdot 1.1789797\ldots$$

Since $S(x) \approx \pi/2$ for x near 0, we see that an "overshoot" by approximately 17.9% is maintained as $n \to \infty$ (but over smaller and smaller intervals centred at $x = 0$).



2.3 Problems

Exercise 1

Find the Fourier coefficients corresponding to the function

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases} \qquad \textit{period} = 10$$

Possible solutions are:

1.

$$a_n = \begin{cases} 3 & n = 0 \\ 0 & n = 1, 2, \dots \end{cases}$$

$$b_n = \begin{cases} \frac{6}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

2.

$$a_n = 0$$

$$b_n = \begin{cases} \frac{6}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

3.

$$a_n = \begin{cases} \frac{6}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$b_n = \begin{cases} 3 & n = 0 \\ 0 & n = 1, 2, \ldots \end{cases}$$

Yes, that's right.

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t dt = \frac{2}{10}\int_0^5 3\cos n\frac{2\pi}{10}t dt = \frac{3}{\pi n}\sin n\pi = 0 \quad n \neq 0$$

$$a_0 = \frac{2}{10}\int_0^5 3dt = 3$$

$$b_n = \frac{2}{10}\int_0^5 3\sin n\frac{2\pi}{10}t dt = \frac{3}{\pi n}(1 - \cos n\pi) = \begin{cases} \frac{6}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$f \sim \frac{3}{2} + \frac{6}{\pi}\left(\sin\frac{\pi}{5}x + \frac{1}{3}\sin\frac{3\pi}{5}x + \ldots\right)$$

Sorry. Please try again.
Sorry. Please try again.

Exercise 2
How should

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases} \qquad \textit{period} = 10$$

be defined at $x = -5, x = 0$ and $x = 5$ in order that the Fourier series will converge to $f(x)$ for $-5 \leq x \leq 5$?

Possible solutions are:

1.

$$f(x) = \begin{cases} 0 & x = -5 \\ 0 & -5 < x < 0 \\ 3/2 & x = 0 \\ 3 & 0 < x < 5 \\ 3 & x = 5 \end{cases}$$

2.

$$f(x) = \begin{cases} 3/2 & x = -5 \\ 0 & -5 < x < 0 \\ 3/2 & x = 0 \\ 3 & 0 < x < 5 \\ 3/2 & x = 5 \end{cases}$$

3.

$$f(x) = \begin{cases} 3 & x = -5 \\ 0 & -5 < x < 0 \\ 3/2 & x = 0 \\ 3 & 0 < x < 5 \\ 0 & x = 5 \end{cases}$$

Sorry. Please try again.
Yes, that's right.

$$\frac{1}{2}(f(-5+) + f(-5-)) = \frac{3}{2}$$

$$\frac{1}{2}(f(-0+) + f(-0-)) = \frac{3}{2}$$

$$\frac{1}{2}(f(5+) + f(5-)) = \frac{3}{2}$$

Sorry. Please try again.

Exercise 3
Expand $f(x) = |x|$, $-\pi < x < \pi$ in a Fourier series if the period is 2π

Possible solutions are:

1.
$$f \sim \frac{\pi}{2} - \frac{4}{\pi}(\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \ldots)$$

2.
$$f \sim \frac{\pi}{2} - \frac{4}{\pi}(\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \frac{\cos 6x}{6^2} + \ldots)$$

3.
$$f \sim \frac{\pi}{2} - \frac{4}{\pi}(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots)$$

Sorry. Please try again.
Sorry. Please try again.
Yes, that's right.

$$b_n = 0 \qquad (f \text{ is even})$$

$$a_n = \frac{1}{\pi} 2 \int_0^{\pi} x \cos(nx) dx = \begin{cases} \pi & n = 0 \\ 0 & n \text{ even} \\ \frac{1}{\pi}\frac{-4}{n^2} & n \text{ odd} \end{cases}$$

Exercise 4 (Orthogonality conditions)
Let $n, m \in \mathbb{N}$. Evaluate the following integrals:

$$\int_{-\pi}^{\pi} e^{imt} e^{-int} dt$$

$$\int_{-\pi}^{\pi} \cos(mt) \cos(nt) dt, \qquad \int_{-\pi}^{\pi} \sin(mt) \sin(nt) dt$$

$$\int_{-\pi}^{\pi} \sin(mt) \cos(nt) dt$$

Possible solutions are

1.

$$\int_{-\pi}^{\pi} e^{imt} e^{-int} dt = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mt)\sin(nt)dt = \int_{-\pi}^{\pi} \cos(mt)\cos(nt)dt = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mt)\cos(nt)dt = 0$$

2.

$$\int_{-\pi}^{\pi} e^{imt} e^{-int} dt = 0$$

$$\int_{-\pi}^{\pi} \sin(mt)\sin(nt)dt = \int_{-\pi}^{\pi} \cos(mt)\cos(nt)dt = 0$$

$$\int_{-\pi}^{\pi} \sin(mt)\cos(nt)dt = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

3.

$$\int_{-\pi}^{\pi} e^{imt} e^{-int} dt = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mt)\sin(nt)dt = \int_{-\pi}^{\pi} \cos(mt)\cos(nt)dt = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mt)\cos(nt)dt = 0$$

Yes, that's right.

$$\int_{-\pi}^{\pi} e^{imt}e^{-int}dt = \int_{-\pi}^{\pi} \cos(m-n)tdt + i\int_{-\pi}^{\pi} \sin(m-n)tdt$$
$$= \begin{cases} \frac{1}{m-n}\sin((m-n)t)\Big|_{-\pi}^{\pi} - \frac{i}{m-n}\cos((m-n)t)\Big|_{-\pi}^{\pi} = 0 & m \neq n \\ 2\pi & m = n \end{cases}$$

Since

$$2\pi\delta_{mn} = \int_{\pi}^{\pi} e^{imt}e^{-int}dt =$$
$$= \int_{-\pi}^{\pi}(\cos(mt)\cos(nt) + \sin(mt)\sin(nt))dt + i\int_{-\pi}^{\pi}(\cos(mt)\sin(nt) - \cos(mt)\sin(nt))dt$$

and

$$0 = \int_{\pi}^{\pi} e^{imt}e^{+int}dt =$$
$$= \int_{-\pi}^{\pi}(\cos(mt)\cos(nt) - \sin(mt)\sin(nt))dt + i\int_{-\pi}^{\pi}(\cos(mt)\sin(nt) + \cos(mt)\sin(nt))dt$$

we get

$$2\pi\delta_{mn} = 2\int_{-\pi}^{\pi}(\cos(mt)\cos(nt))dt + 2i\int_{-\pi}^{\pi}(\cos(nt)\sin(mt))dt$$

Therefore

$$0 = \int_{-\pi}^{\pi}(\cos(nt)\sin(mt))dt$$

and

$$\pi\delta_{mn} = \int_{-\pi}^{\pi}(\cos(mt)\cos(nt))dt$$

Integration by parts

$$\pi\delta_{mn} = \int_{-\pi}^{\pi}(\sin(mt)\sin(nt))dt$$

Sorry. Please try again.
Sorry. Please try again.

Exercise 5
Find the Fourier representation (period 2π) of

$$f(x) = \sin(3x) + \cos^2(x)$$

Possible solutions are:

1.
$$f \sim \sin(3x) + \cos^2(x)$$

2.
$$f \sim \frac{1}{2} + \sin(3x) + \cos(x) + \frac{1}{2}\cos^2(x) + \frac{1}{3}\cos^3(x) + \dots$$

3.
$$f \sim \frac{1}{2} + \sin(3x) + \frac{1}{2}\cos(2x)$$

Sorry. Please try again.
Sorry. Please try again.
Yes, that's right.

$$\begin{aligned}\frac{1}{2}\cos(2x) + \frac{1}{2} &= \frac{1}{2}(\cos^2(x) - \sin^2(x)) + \frac{1}{2} \\ &= \frac{1}{2}(2\cos^2(x) - 1) + \frac{1}{2} = \cos^2(x)\end{aligned}$$

Exercise 6
Let f be piecewise continuous. Evaluate

$$\lim_{m\to\infty} \int_{-\pi}^{\pi} f(x)\sin(m + \frac{1}{2})x dx$$

Possible solutions are:

1.
$$\lim_{m\to\infty} \int_{-\pi}^{\pi} f(x)\sin(m + \frac{1}{2})x dx = \frac{\pi}{2}$$

2.

$$\lim_{m\to\infty}\int_{-\pi}^{\pi} f(x)\sin(m+\frac{1}{2})x dx = 2\pi$$

3.

$$\lim_{m\to\infty}\int_{-\pi}^{\pi} f(x)\sin(m+\frac{1}{2})x dx = 0$$

Sorry. Please try again.
Sorry. Please try again.
Yes, that's right.
Since

$$\sin((m+1/2)x) = \sin(mx)\cos(x/2) + \cos(mx)\sin(x/2)$$

It follows from the Riemann lemma

$$\begin{aligned}\lim_{m\to\infty}\int_{-\pi}^{\pi} f(x)\sin(m+\frac{1}{2})x dx &= \lim_{m\to\infty}(\int_{-\pi}^{\pi}(f(x)\cos(x/2))\sin(mx)dx + \int_{-\pi}^{\pi}(f(x)\sin(x/2))\cos(mx)d\\ &= 0\end{aligned}$$

Exercise 7
From the Fourier series

$$\sum_{k=1}^{\infty}\frac{\sin kx}{x} = \frac{\pi - x}{2}, \quad 0 < x < 2\pi$$

the following equation can be obtained.

$$S(x) = \sum_{n=1}^{\infty}\frac{\cos nx}{n^2} = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12} \qquad 0 \le x \le 2\pi$$

It is true that

1. $S(x)$ has a overshoot near $x = 2\pi$ by approximately 17.9%.
2. $S(x)$ has a overshoot near $x = 0$ by approximately 17.9%.

3. $S(x)$ has no overshoot.

Sorry. Please try again.
Sorry. Please try again.
Yes, that's right.
Since $S(x)$ converges uniformly on $\mathbb{R}$, and $\frac{\cos nx}{n^2}$ is continuous, it follows that $S(x)$ is continuous. Because there is no jump of discontinuity, $S(x)$ has no overshoot.

3 Fourier transforms

A Fourier series can sometimes be used to represent a function over an interval. If a function is defined over the entire real line, it may still have a Fourier series representation if it is periodic. If it is not periodic, then it cannot be represented by a Fourier series for all x. In such case we may still be able to represent the function in terms of sines and cosines, except that now the Fourier series becomes a Fourier integral.

The motivation comes from formally considering Fourier series for functions of period $2T$ and letting T tend to infinity.

Suppose

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\frac{\pi}{T}x}$$

and

$$c_n = \frac{1}{2T}\int_{-T}^{T} e^{-in\frac{\pi}{T}t} f(t)dt$$

Now, set

$$\omega_n = \frac{n\pi}{T} \quad \text{and} \quad \Delta\omega = \omega_n - \omega_{n-1} = \frac{\pi}{T}$$

and insert the integral formula for the Fourier coefficients:

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(e^{i\omega_n x}\int_{-T}^{T} e^{-i\omega_n t} f(t)dt\right)\Delta\omega$$

The summation resembles a Riemann sum for a definite integral, and in the limit $T \to \infty$ $(\Delta\omega \to 0)$ we might get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{i\omega x}\int_{-T}^{T} e^{-i\omega t} f(t)dt\right)d\omega \qquad x \in \mathbb{R}$$

This informal reasoning suggest the following definition:

Definition 6 (Fourier Transforms)
A function $F(\omega)$ is called the Fourier transform *of $f(x)$, if*

$$F(\omega) := \mathcal{F}\{f(t)\} := \int_{-\infty}^{\infty} e^{-i\omega t} f(t)dt \tag{10}$$

exist.

$$\mathcal{F}^{-1}\{F(\omega)\} := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x} F(\omega)d\omega \tag{11}$$

is called the inverse Fourier transform *of $F(\omega)$.*

The Fourier transform of f is therefore a function $\mathcal{F}\{f(t)\}$ of the new variable ω. This function, evaluated at ω, is $F(\omega)$.

Remark 5
The constants 1 and $1/2\pi$ preceding the integral signs in (10) and (11)could be replaced by any two constants whose product is $1/2\pi$.

Example 7
The Fourier transform of f given by

$$f(t) := \begin{cases} 1 & |t| < 1 \\ 0 & |t| > 1 \end{cases}$$

is

$$\mathcal{F}\{f(t)\} = \int_{-1}^{1} e^{-i\omega t}dt = \begin{cases} 2\frac{\sin\omega}{\omega} & \omega \neq 0 \\ 2 & \omega = 0 \end{cases}$$

The inverse Fourier transform computes to

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty} e^{i\omega x}\frac{\sin\omega}{\omega}d\omega &= \frac{2}{\pi}\int_0^{\infty}\cos\omega x\frac{\sin\omega}{\omega}d\omega \\ &= \frac{1}{\pi}\int_0^{\infty}\frac{\sin\omega(x+1)}{\omega}d\omega - \frac{1}{\pi}\int_0^{\infty}\frac{\sin\omega(x-1)}{\omega}d\omega \\ &= \begin{cases} 1 & |x|<1 \\ 1/2 & |x|=1 \\ 0 & |x|>1 \end{cases}\end{aligned}$$

Theorem 8 (The Fourier integral)

1. *If $f(x)$ and $f'(x)$ are piecewise continuous in every finite interval*
2. *and $\int_{-\infty}^{\infty}|f(x)|dx$ converges, i.e $f(x)$ is absolutely integrable in $(-\infty,\infty)$*

Then

$$\frac{1}{2}[f(x-)+f(x+)] = \frac{1}{2\pi}\lim_{\Omega\to\infty}\int_{-\Omega}^{\Omega} e^{i\omega x}\left(\int_{-\infty}^{\infty} e^{-i\omega t}f(t)dt\right)d\omega \qquad (12)$$

Remark 6

The above conditions are sufficient but not necessary. The similarity with corresponding results for Fourier series is apparent.

We will now develop some properties of the Fourier transform:

Linearity

If $\alpha,\beta\in\mathbb{C}$, then

$$\mathcal{F}\{\alpha f(t)+\beta g(t)\} = \alpha\mathcal{F}\{f(t)\}+\beta\mathcal{F}\{g(t)\} = \alpha F(\omega)+\beta G(\omega)$$

provided the Fourier transform of $f(t)$ and $g(t)$ exist.

Scaling

If $\mathcal{F}\{f(t)\}=F(\omega)$ and $c\in\mathbb{R}$, then

$$\mathcal{F}\{ct\} = \frac{1}{|c|}F(\frac{\omega}{c}), \quad c\neq 0$$

Time shifting

If $\mathcal{F}\{f(t)\} = F(\omega)$ and $t_0 \in \mathbb{R}$, then

$$\mathcal{F}\{f(t-t_0)\}) = e^{-i\omega t_0}F(\omega), \quad t_0 \in \mathbb{R}$$

Proof.

$$\begin{aligned}\mathcal{F}\{f(t-t_0\} &= \int_{-\infty}^{\infty} f(t-t_0)e^{-i\omega t}dt \\ &= e^{-i\omega t_0}\int_{-\infty}^{\infty} f(u)e^{-i\omega(u)}du\end{aligned}$$

□

Frequency shifting

If $\mathcal{F}\{f(t)\} = F(\omega)$ and $\omega_0 \in \mathbb{R}$, then

$$\mathcal{F}\{e^{i\omega_0 t}f(t)\} = F(\omega-\omega_0), \quad \omega_0 \in \mathbb{R}$$

Proof.

$$\mathcal{F}\{e^{i\omega_0 t}f(t)\} = \int_{-\infty}^{\infty} e^{i\omega_0 t}f(t)e^{-i\omega t}dt = F(\omega-\omega_0)$$

□

Symmetry

If $\mathcal{F}\{f(t)\} = F(\omega)$, then

$$\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$$

Proof. Use the formula for the inverse Fourier transform

$$f(t) = \mathcal{F}^{-1}\{F(\omega\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)e^{ixt}dx$$

Then

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(x)e^{-ix\omega}dx = \int_{-\infty}^{\infty} F(t)e^{-it\omega}dt = \mathcal{F}\{F(t)\}$$

□

Modulation

If $\mathcal{F}\{f(t)\} = F(\omega)$ and $\omega_0 \in \mathbb{R}$, then

$$\mathcal{F}\{f(t)\cos(\omega_0 t)\} = \frac{1}{2}[F(\omega+\omega_0) + F(\omega-\omega_0)]$$
$$\mathcal{F}\{f(t)\sin(\omega_0 t)\} = \frac{1}{2}[F(\omega+\omega_0) - F(\omega-\omega_0)]$$

Proof. Use the frequency-shifting theorem to get

$$\begin{aligned}\mathcal{F}\{f(t)\cos(\omega_0 t)\} &= \frac{1}{2}[\mathcal{F}\{e^{i\omega_0 t}f(t)\} + \mathcal{F}\{e^{-i\omega_0 t}f(t)\}] \\ &= \frac{1}{2}[F(\omega+\omega_0) + F(\omega-\omega_0)]\end{aligned}$$

□

Differentiation in time

Let $n \in \mathbb{N}$ and suppose that $f^{(n)}$ is piecewise continuous. Assume that $\lim_{t\to\infty} f^{(k)}(t) = \lim_{t\to-\infty} f^{(k)}(t) = 0$. Then

$$\mathcal{F}\{f^{(n)}(t)\} = (i\omega)^n F(\omega)$$

In particular

$$\mathcal{F}\{f'(t)\} = i\omega F(\omega)$$

and

$$\mathcal{F}\{f''(t)\} = -\omega^2 F(\omega)$$

Proof. Assume $n = 1$. The general case can be proved by induction.

$$\int_{-\infty}^{\infty} f'(t)e^{-i\omega t}dt = f(t)e^{-i\omega t}\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)(-i\omega)e^{-i\omega t}dt = i\omega F(\omega)$$

□

Example 8

Suppose we want

$$\mathcal{F}\{f(t)\} = \mathcal{F}\{e^{-t^2}\}$$

We apply differentiation in time to get

$$\mathcal{F}\{tf(t)\} = iF'(\omega)$$

We can also integrate by parts to get

$$\mathcal{F}\{tf(t)\} = -\frac{i\omega}{2}F(\omega)$$

Then we have

$$F'(\omega) = -\frac{\omega}{2}F(\omega) \qquad F(0) = \sqrt{(\pi)}$$

Solving this equation, we get

$$\mathcal{F}\{e^{-t^2}\} = \sqrt{\pi}e^{-\frac{\omega^2}{4}}$$

Example 9

Suppose we want to solve

$$y' - 4y = H(t)e^{-4t}$$

$H(t)$ *is given by*

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Apply the Fourier transform to the differential equation to get

$$\mathcal{F}\{y'\} - 4\mathcal{F}\{y\} = \mathcal{F}\{H(t)e^{-4t}\} = \frac{1}{4 + i\omega}$$

Setting $\mathcal{F}\{y(t)\} = Y(\omega)$, *we have*

$$i\omega Y(\omega) - 4Y(\omega) = \frac{1}{4 + i\omega}$$

Then

$$Y(w) = \frac{1}{(4 + i\omega)(-4 + i\omega)} = \frac{-1}{16 + \omega^2} = \mathcal{F}\{-\frac{1}{8}e^{-4|t|}\}$$

From the last equation, we get

$$y(t) = \mathcal{F}^{-1}Y(\omega) = -\frac{1}{8}e^{-4|t|}$$

Frequency differentiation

Let $n \in \mathbb{N}$ and suppose that f is piecewise continuous. Then

$$\mathcal{F}\{t^n f(t)\} = i^n F^{(n)}(\omega)$$

In particular

$$\mathcal{F}\{tf(t)\} = iF'(\omega)$$

and

$$\mathcal{F}\{t^2 f(t)\} = -F''(\omega)$$

Proof. We will prove the theorem for $n = 1$. The argument for larger n is repetition of this.

$$F'(\omega) = \int_{-\infty}^{\infty} [f(t)e^{-i\omega t}]dt = -i\int_{-\infty}^{\infty} (tf(t))e^{-i\omega t}dt = -i\mathcal{F}\{tf(t)\}$$

$\square$

Convolution

Definition 7 (The convolution (faltung))
If f and g both have Fourier transforms, then the convolution (faltung) $f * g$ of the functions f and g is defined by

$$f * g = \int_{-\infty}^{\infty} f(u)g(x-u)du \tag{13}$$

Theorem 9 (The convolution theorem)
The Fourier transform of the convolution of $f(x)$ and $g(x)$ is equal to the product of the Fourier transforms of $f(x)$ and $g(x)$.

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\}\mathcal{F}\{g\} \qquad \textit{(time convolution)}$$

and

$$\mathcal{F}\{f(t)g(t)\} = \frac{1}{2\pi}[F * G](\omega) \qquad \textit{(frequency convolution)}$$

Proof. If $\int_{-\infty}^{\infty} |f| dt, \int_{-\infty}^{\infty} |g| dt < \infty$ it follows for the time convolution formula

$$
\begin{aligned}
\mathcal{F}\{f\}\mathcal{F}\{g\} &= \int_{-\infty}^{\infty} e^{-i\omega t} G(\omega) f(t) dt \\
&= \int_{-\infty}^{\infty} e^{-i\omega t} \left(\int_{-\infty}^{\infty} e^{-i\omega\tau} g(\tau) d\tau \right) f(t) dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega(t+\tau)} g(\tau) f(t) d\tau dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega u} g(\tau) f(u-\tau) d\tau dt \\
&= \int_{-\infty}^{\infty} e^{-i\omega u} \left(\int_{-\infty}^{\infty} g(\tau) f(u-\tau) d\tau \right) du
\end{aligned}
$$

A proof of the frequency convolution goes analog. □

The Fourier transform of the Dirac delta function

Some problems involve the concept of an impulse, which may be intuitively thought of as a force of very large magnitude impacting just for an instant. We can model this idea mathematically as follows:

$$
\delta_\epsilon(t) = \begin{cases} 0 & -\infty < t < 0 \\ 1/\epsilon & 0 < t < \epsilon \\ 0 & \epsilon < t < \infty \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta_\epsilon(t) = 1
$$

As ϵ is chosen smaller, the duration of this pulse tends to zero while its amplitudes increases without bound. This lead us to define

$$
\delta(t) := \lim_{\epsilon \to 0} \delta_\epsilon(t)
$$

Strictly speaking, $\delta(t)$ is not really a function in the conventional sense, but it is a quantity called <u>distribution</u>. For historical reasons it is called the <u>Dirac delta function</u> after the physicist P. A. M. Dirac. The delta function has the fundamental property:

Definition 8 (Dirac delta function)

$$
\begin{aligned}
\int_{-\infty}^{\infty} g(t)\delta(t-t_0) dt := \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} g(t)\delta_\epsilon(t-t_0) dt = \\
= g(t_0), \qquad \textit{if } f \textit{ is continous in } t_0
\end{aligned}
$$

Remark 7
Therefore the Fourier transform of the delta function yields 1.

$$\mathcal{F}\{\delta(t)\} = 1$$

The sampling theorem

A function $f(t)$ is called band-limited if its Fourier transform is only nonzero on an interval of finite length. This means for some L, $F(\omega) = 0$ if $|\omega| > L$.

Begin with the integral for the inverse Fourier transform.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-L}^{L} F(\omega)e^{i\omega t}d\omega$$

The complex Fourier series for $F(\omega)$ on $[-L, L]$ is given by

$$F(\omega) = \sum_{n=-\infty}^{\infty} c_n e^{n\pi i\omega/L}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} F(\omega)e^{-n\pi i\omega/L}d\omega$$

Now compare this equations to conclude that

$$c_n = \frac{\pi}{L}f\left(-\frac{n\pi}{L}\right)$$

and

$$F(\omega) = \sum_{n=-\infty}^{\infty} \frac{\pi}{L}f\left(\frac{n\pi}{L}\right)e^{n\pi i\omega/L}$$

Substitute this series for $F(\omega)$ in $f(t)$ to get

$$f(t) = \frac{1}{2\pi}\frac{\pi}{L}\int_{-L}^{L}\sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{L}\right)e^{-n\pi i\omega/L}e^{i\omega t}d\omega$$

Interchanging the summation and the series, we get

Theorem 10 (The sampling theorem)

$$
\begin{aligned}
f(t) &= \frac{1}{2L} \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) \int_{-L}^{L} e^{i\omega(t-n\pi/L)} d\omega \\
&= \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) \frac{\sin(Lt-n\pi)}{Lt-n\pi}
\end{aligned}
$$

This means that $f(t)$ is known for all t if just the function values $f(n\pi/L)$ are known for integer values of n. That is, if we sample the signal (function) and determine its values at $0, \pm\pi/L, \pm 2\pi/L, \dots$, then the entire signal can be reconstructed.

3.1 FFT

Often we are interested in properties of a function f, knowing only measured values of f at equally spaced time intervals

$$t_k = k\Delta t, \quad k \in \mathbb{Z}, \Delta t > 0$$

If this discrete function f has the period $T = N\Delta t$, then f is described by the vector

$$
\mathbf{y} := \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{pmatrix} = \begin{pmatrix} f(0) \\ f(\Delta t) \\ \vdots \\ f((N-1)\Delta t) \end{pmatrix}
$$

Definition 9 (Discrete Fourier coefficient)
Assume $T = 2\pi$ then the Fourier coefficient of y is defined

$$c_k := \frac{1}{N} \sum_{j=0}^{N-1} y_j e^{-kj\frac{2\pi i}{N}}, \quad k = 0, 1, \dots, N-1$$

Definition 10 (Discrete Fourier transform (DFT))
The mapping $F : \mathbb{C}^N \to \mathbb{C}^N$, defined by

$$
F(y) = c, \quad c = \begin{pmatrix} c_0 \\ \vdots \\ c_{N-1} \end{pmatrix} \in \mathbb{C}^N, \quad y = \begin{pmatrix} y_0 \\ \vdots \\ y_{N-1} \end{pmatrix} \in \mathbb{C}^N
$$

with

$$
\begin{aligned}
c_k &:= \frac{1}{N}\sum_{j=0}^{N-1} y_j \overline{w_N^{kj}}, \qquad k = 0, 1, \dots, N-1 \\
w_N &:= e^{\frac{2\pi i}{N}}
\end{aligned} \tag{14}
$$

is called the <u>*discrete Fourier transform (DFT)*</u>*.*
If we use the <u>$N \times N$ *Fourier-Matrix*</u>

$$
F_n := \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & w_N & w_N^2 & \cdots & w_N^{N-1} \\
1 & w_N^2 & w_N^4 & \cdots & w_N^{2(N-1)} \\
\vdots & \vdots & \vdots & \vdots & \\
1 & w_N^{N-1} & w_N^{2(N-1)} & \cdots & w_N^{(N-1)^2}
\end{pmatrix}
$$

then we can write (14) as

$$
c = \frac{1}{N} F_N y
$$

Theorem 11
It follows, that

$$
F_N \overline{F}_N = \overline{F}_N F_N = NE
$$

i.e.

$$
F_N^{-1} = \frac{1}{N}\overline{F}_N
$$

Proof.

$$
\sum_{j=0}^{N-1} w_N^{kj}\overline{w_N^{jl}} = \sum_{j=0}^{N-1} w_N^{(k-l)j} = \begin{cases} N, & l = k \\ 0, & l \neq k \end{cases}
$$

This is because $z := w_N^{k-l}$ is a Nth root of Unity and $z \neq 1$ if $k \neq l$

$$
z^{N-1} + z^{N-2} + \ldots + z + 1 = \frac{z^N - 1}{z - 1} = 0 \qquad z \neq 1
$$

$\square$

Definition 11 (Inverse discrete Fourier transform (IDFT))
The inverse mapping $y = F_N c$ is called the inverse discrete Fourier transform (IDFT)

$$y_j = \sum_{k=0}^{N-1} c_k w_N^{jk}, \qquad j = 0, 1, \ldots, N-1 \tag{15}$$

Some properties of the DFT are:

Linearity

$$\alpha y + \beta z \xrightarrow{\text{DFT}} \alpha c + \beta d$$

Parseval

$$\sum_{k=0}^{N-1} |c_k|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |y_k|^2$$

Theorem 12 (Fast Fourier Transform (FFT))
IF N is even (N=2M), then $y = F_N c$ (and analog $c = \frac{1}{N}\overline{F_N} y$) can be put down to two discrete transforms.

We divide c in its odd and even indices

$$e := (c_0, c_2, \ldots, c_{N-2})^T \in \mathbb{C}^M$$

and

$$o := (c_1, c_3, \ldots, c_{N-1})^T \in \mathbb{C}^M$$

$$y_k = \sum_{j=0}^{N-1} w_N^{jk} c_j = \sum_{j=0}^{M-1} (w_N^2)^{kj} e_j + w^k \sum_{j=0}^{M-1} (w_N^2)^{kj} o_j \quad k = 0, 1, \ldots, N-1$$

y is splitted in

$$a := (y_0, y_1, \ldots, y_{M-1})^T \in \mathbb{C}^M$$

and

$$b := (y_M, y_{M+1}, \ldots, y_{N-1})^T \in \mathbb{C}^M$$

It follows ($w_N^{k+M} = -w_N^k$) that

$$a_k = \sum_{j=0}^{M-1} (w_N^2)^{kj} e_j + w^k \sum_{j=0}^{M-1} (w_N^2)^{kj} o_j \quad k = 0, 1, \ldots, M-1$$
$$b_k = \sum_{j=0}^{M-1} (w_N^2)^{kj} e_j - w^k \sum_{j=0}^{M-1} (w_N^2)^{kj} o_j \quad k = 0, 1, \ldots, M-1$$

w_N^2 is an Mth root of unity, so the above sums describe two IDFT

$$a = F_M e + \mathrm{Diag}(1, w_M, \ldots, w_M^{M-1}) F_M o$$
$$b = F_M e - \mathrm{Diag}(1, w_M, \ldots, w_M^{M-1}) F_M o$$

In order to perform a Fourier transform of length N, one need to do two Fourier transforms $F_M e$ and $F_M o$ of length M on the even and odd elements. We now have two transforms which take less time to work out. The two sub-transforms can then be combined with the appropriate factor w^k to give the IDFT. Applying this recursively leads to the algorithm of the Fast Fourier transform (FFT).

```
/*
  x and y are real and imaginary arrays of 2^m points.
  dir = 1 gives forward transform
  dir = -1 gives reverse transform
*/


FFT(int dir, int m, double *x, double *y)
{
  int n,i,i1,j,k,i2,l,l1,l2;
  double c1,c2,tx,ty,t1,t2,u1,u2,z;

  /* Number of points */
  n = 1;
  for (i=0;i<m;i++)
    n *= 2;

  /* Bit reversal */
```

```
i2 = n >> 1;
j = 0;
for (i=0;i < n-1; i++) {
  if (i < j) {
    tx = x[i];
    ty = y[i];
    x[i] = x[j];
    y[i] = y[j];
    x[j] = tx;
    y[j] = ty;
  }
  k = i2;
  while (k <= j) {
    j -= k;
    k >>= 1;
  }
  j += k;
}

for (i=0;i < n; i++) {
  printf("x[%i] = %f   y[%i] = %f\n", i, x[i], i, y[i]);
}
printf("-------------------------\n");

/* compute the FFT */
c1 = -1.0;
c2 = 0.0;
l2 = 1;
for (l=0;l<m;l++) {
  l1 = l2;
  l2 <<= 1;
  u1 = 1.0;
  u2 = 0.0;
  for (j=0;j<l1;j++) {
    for (i=j;i<n;i+=l2) {
      i1 = i + l1;
      t1 = u1 * x[i1] - u2 * y[i1];
      t2 = u1 * y[i1] + u2 * x[i1];
```

```
        x[i1] = x[i] - t1;
        y[i1] = y[i] -t2;
        x[i]  += t1;
        y[i]  += t2;
      }
      z =  u1 * c1 -u2 * c2;
      u2 = u1 * c2 + u2 * c1;
      u1 = z;
    }
    c2 = sqrt((1.0 - c1) / 2.0);
    if (dir == 1)
        c2 = -c2;
    c1 = sqrt((1.0 + c1) / 2.0);
  }

  /* scaling for forward transform */
  if (dir == 1) {
    for (i=0;i<n;i++) {
      x[i] /= n;
      y[i] /= n;
    }
  }
}
```

3.2 Problems

Exercise 8

The Heaviside function $H(t)$ is given by

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Find the Fourier transform of

$$f(t) = H(t)e^{-at}$$

Possible solutions are

1.

$$F(\omega) = \frac{1}{a + i\omega}$$

2.

$$F(\omega) = \frac{1}{a+\omega}$$

3.

$$F(\omega) = \frac{1}{\omega + ia}$$

Yes, that's right.
The Fourier transform is

$$\begin{aligned} F(\omega) &= \int_0^\infty e^{-at} e^{-i\omega t} dt \\ &= \frac{-1}{a+i\omega} e^{-(a+i\omega)t} \Big|_0^\infty \\ &= \frac{1}{a+i\omega} \end{aligned}$$

Sorry. Please try again.
Sorry. Please try again.

Exercise 9
Determine the value of

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{2i\omega}}{5+i\omega} e^{i\omega t} d\omega$$

Possible solutions are:

1.

$$H(t+2)e^{-5(t+2)}$$

2.

$$H(t)e^{2it}$$

3.

$$H(e^{-5(t+2)}))$$

Yes, that's right.

$$\mathcal{F}^{-1}\{\frac{1}{5+i\omega}\} = f(t) = H(T)e^{-5t}$$

Applying the time time-shifting theorem on $f(t+2)$

$$\mathcal{F}\{f(t+2)\} = \frac{e^{2i\omega}}{5+i\omega}$$

Therefore

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{2i\omega}}{5+i\omega}e^{i\omega t}d\omega = f(t+2) = H(t+2)e^{-5(t+2)}$$

Sorry. Please try again.
Sorry. Please try again.

Exercise 10
Use the Fourier transform to find one solution $y(t)$ of

$$y(t)'' + 3y(t)' + 2y(t) = 0$$

1.

$$y(t) = \begin{cases} e^{-t} - e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

2.

$$y(t) = \begin{cases} e^{t} - e^{2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

3.

$$y(t) = \begin{cases} e^{t} + e^{2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Yes, that's right.
Apply the Fourier transform on the differential equation

$$-\omega^2 Y(\omega) + 3i\omega Y(\omega) + 2Y(\omega) = 0$$

$$Y(\omega) = \frac{1}{-\omega^2 + 3i\omega + 2}$$

Therefore

$$y(t) = \mathcal{F}^{-1}\{\frac{1}{-\omega^2 + 3i\omega + 2}\}$$

Factoring

$$-\omega^2 + 3i\omega + 2 = (2 + i\omega)(1 + i\omega)$$

Then

$$y(t) = \mathcal{F}^{-1}\{\frac{1}{2 + i\omega}\frac{1}{1 + i\omega}\} = H(t)e^{-2t} * H(t)e^{-t}$$

Compute this convolution

$$H(t)e^{-2t} * H(t)e^{-t} = \begin{cases} e^{-t}\int_0^t e^{-\tau} d\tau & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Finally

$$y(t) = \begin{cases} e^{-t} - e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Sorry. Please try again.
Sorry. Please try again.

Exercise 11 (Convolution theorem)
The Fourier transform of $y(x), g(x)$ and $r(x)$ are denoted by $Y(\omega), G(\omega)$ and $R(\omega)$. Solve for y(x) the integral equation

$$y(x) = g(x) + \int_{-\infty}^{\infty} y(u)r(x-u)du$$

Possible solutions are:

1.
$$y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) * R(\omega) e^{i\omega x} d\omega$$

2.
$$y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - R(\omega)}{G(\omega)} e^{i\omega x} d\omega$$

3.
$$y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G(\omega)}{1 - R(\omega)} e^{i\omega x} d\omega$$

Sorry. Please try again.
Sorry. Please try again.
Yes, that's right.
Taking the Fourier transforms of both sides of the given integral equation, we have by the convolution theorem

$$Y(\omega) = G(\omega) + Y(\omega) R(\omega)$$

or

$$Y(\omega) = \frac{G(\omega)}{1 - R(\omega)}$$

Then

$$y(x) = \mathcal{F}^{-1}\{Y(\omega)\} = \frac{G(\omega)}{1 - R(\omega)}$$

Exercise 12
The Fourier matrix F_4 is

1.
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & 0 \\ 1 & i & 0 & i \end{pmatrix}$$

2.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & i \end{pmatrix}$$

3.

$$\begin{pmatrix} 1 & i & -1 & -i \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & i \end{pmatrix}$$

Sorry. Please try again.
Yes, that's right.

$$\begin{aligned} w_4 = e^{i\pi/2} = i = w_4^5 = w_4^9 \\ w_4^2 = e^{i\pi/2} = -1 = w_4^6 \\ w_4^3 = e^{i\pi/2} = -i = w_4^7 \\ w_4^4 = e^{i\pi/2} = 1 = w_4^8 \end{aligned}$$

$$F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & i \end{pmatrix}$$

Sorry. Please try again.

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