A Mach-uniform preconditioner for incompressible and subsonic flows

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SUMMARY

In this study, a novel Mach-uniform preconditioning method is developed for the solution of Euler equations at low subsonic and incompressible flow conditions. In contrast to the methods developed earlier in which the conservation of mass equation is preconditioned, in the present method, the conservation of energy equation is preconditioned, which enforces the divergence free constraint on the velocity field even at the limiting case of incompressible, zero Mach number flows. Despite most preconditioners, the proposed Mach-uniform preconditioning method does not have a singularity point at zero Mach number. The preconditioned system of equations preserves the strong conservation form of Euler equations for compressible flows and recovers the artificial compressibility equations in the case of zero Mach number. A two-dimensional Euler solver is developed for validation and performance evaluation of the present formulation for a wide range of Mach number flows. The validation cases studied show the convergence acceleration, stability, and accuracy of the present Mach-uniform preconditioner in comparison to the non-preconditioned compressible flow solutions. The convergence acceleration obtained with the present formulation is similar to those of the well-known preconditioned system of equations for low subsonic flows and to those of the artificial compressibility method for incompressible flows. Copyright © 2013 John Wiley & Sons, Ltd.

Received 26 November 2012; Revised 26 June 2013; Accepted 30 July 2013

KEY WORDS: preconditioning; artificial compressibility method; Mach-uniform accuracy

1. INTRODUCTION

The inviscid flows are governed by the Euler equations. Although the Euler equations are valid for all Mach number flows, the widely used numerical methods developed for the solution of Euler equations suffer from severe stability and accuracy problems for low Mach number flows and, at the limiting case, incompressible flows. There are various approaches for removing these difficulties [1, 2]. The well-known methods are preconditioning of the Euler equations for low Mach number flows and the artificial compressibility method (ACM) for incompressible flows.

1.1. Cancellation problem

The non-dimensional Euler equations have the well-known ‘cancellation problem’ as Mach number goes to zero. The 2-D Euler equations are given by

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0
\]

\[
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} + \frac{\partial (\rho uv)}{\partial y} = 0
\]
The ideal gas equation is used to close the aforementioned system of equations.

\[ p = (\gamma - 1) \left( \rho c_t - \frac{1}{2} \rho (u^2 + v^2) \right) \]

When the Euler equations are non-dimensionalized as follows

\[ \rho^* = \frac{\rho}{\rho_\infty}, \quad u^* = \frac{u}{U_\infty}, \quad x^* = \frac{x}{x_r}, \quad t^* = \frac{tu_\infty}{x_r}, \quad p^* = \frac{p}{\rho_\infty} \]

the non-dimensional conservation of \( x \)-momentum equation becomes

\[ \frac{\partial \rho^* u^*}{\partial t^*} + \frac{\partial (\rho^* u^* u^*)}{\partial x^*} + \frac{\partial (\rho^* u^* v^*)}{\partial y^*} = 0 \]  

It can be shown that the term \( \frac{p_\infty}{\rho_\infty u_\infty} \) is equal to \( \frac{1}{\gamma M_\infty^2} \). As \( M_\infty \) goes to zero, this term goes to infinity, which is known as the ‘cancellation problem’. The cancellation problem still exists even if the velocity is non-dimensionalized with the speed of sound instead. On the other hand, if the pressure is non-dimensionalized with \( \frac{\rho_\infty u_\infty^2}{p_\infty} \), the cancellation problem is now observed in the boundary conditions: The free-stream pressure boundary condition becomes

\[ p^* = \frac{p - p_\infty}{\rho_\infty U_\infty^2} \]

Now, the conservation \( x \)-momentum equation becomes

\[ \frac{\partial \rho^* u^*}{\partial t^*} + \frac{\partial (\rho^* u^* u^*)}{\partial x^*} + \frac{\partial (\rho^* u^* v^*)}{\partial y^*} = 0 \]  

Because the term \( \frac{1}{\gamma M_\infty^2} \) is constant, its derivative drops out. The pressure splitting method also sets the far field pressure boundary condition to zero: \( \frac{p - p_\infty}{\rho_\infty U_\infty^2} = 0 \). The cancellation problem is also observed on the solution of the conservation of energy equation [4], which may be resolved by a similar approach employed for the non-dimensionalization of total energy.

1.2. Preconditioning of Euler equations

Another major difficulty in the solution of the Euler equations reveals itself when the eigenvalues are analyzed. The increase in the ratio between the maximum and the minimum eigenvalues as Mach number decreases is known as the ‘eigenvalue disparity’. The eigenvalue disparity makes the solution of Euler equations for low Mach number flows very stiff. The over-relaxation of the time derivative terms is a well-known numerical solution method against the eigenvalue disparity, from which preconditioning methods are inspired. In preconditioning methods, the time derivative terms in the Euler equations are premultiplied by a matrix to enhance the convergence behavior without altering the steady-state solution. The pioneering research on preconditioning methods is conducted by Turkel [5], and a family of preconditioners are introduced. Several improvements and extensions of preconditioning methods are later developed by Choi and Merkle [6], Weiss and Smith [7], and van Leer [8]. Beside the improvements in formulations, widely used acceleration schemes such as multi-grid approach and implicit time stepping are also applied on preconditioning formulations [9,10]. References [11,12] provide a detailed review of preconditioning methods.
1.3. Artificial compressibility method

Beyond the low Mach number flows, the ACM was originally developed by Chorin [13] for the solution of incompressible flows and has been widely employed in the literature [14–16]. The artificial compressibility form of the Euler equations are [13]

\[
\frac{\partial \hat{p}}{\partial t} + \frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = 0
\]

\[
\frac{\partial \hat{u}}{\partial t} + \frac{\partial (u^2 + p)}{\partial x} + \frac{\partial (uv)}{\partial y} = 0
\]

\[
\frac{\partial \hat{v}}{\partial t} + \frac{\partial (uv)}{\partial x} + \frac{\partial (v^2 + p)}{\partial y} = 0
\]

The equation set given previously differs from the incompressible flow equations by the presence of the \(\frac{\partial \hat{p}}{\partial t}\) term. \(\hat{p}\) is defined as a new variable named as artificial density, and a new relationship between pressure and artificial density is formed. This artificial equation of state is defined as \(p = \hat{p}/\delta\) where \(\delta\) is the artificial compressibility parameter. It is stated by Chorin [13] that for a steady-state solution of the ACM equation set, \(\delta\) appears to be disposable and is analogous to a relaxation parameter. It should be noted that in ACM, the conservation of energy equation is decoupled from the conservation of mass and momentum equations and is excluded from the solution algorithm.

2. MACH-UNIFORM PRECONDITIONING OF EULER EQUATIONS

In the present formulation, the Euler equations are non-dimensionalized similar to the pressure-splitting method, which is also employed in ACM. Furthermore, in addition to pressure, total energy is also split in a similar manner. The present non-dimensional variables are as follows:

\[
\rho^* = \frac{\rho}{\rho_\infty} \quad u^* = \frac{u}{u_\infty} \quad p^* = \frac{p}{\rho_\infty U_\infty^2} - p_r \quad (pe_t)^* = \frac{pe_t}{\rho_\infty U_\infty^2} - (pe_t)_r
\]

\[
x^* = \frac{x}{x_r} \quad t^* = \frac{t u_\infty}{x_r}
\]

The reference values of \(p_r\) and \((pe_t)_r\) are chosen as \(\frac{p_\infty}{\rho_\infty U_\infty^2}\) and \(\frac{(pe_t)_\infty}{\rho_\infty U_\infty^2}\), which are equivalent to \(\frac{1}{\gamma M_\infty^2}\) and \(\frac{1}{\gamma (\gamma - 1) M_\infty^2}\), respectively. The non-dimensional conservation equations then become

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0
\]

\[
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} + \frac{\partial (\rho uv)}{\partial y} = 0
\]

\[
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho uv)}{\partial x} + \frac{\partial (\rho v^2 + p)}{\partial y} = 0
\]

\[
\frac{\partial (pe_t)}{\partial t} + \frac{\partial \left( u \left( pe_t + p + \frac{1}{(\gamma - 1)M_\infty^2} \right) \right)}{\partial x} + \frac{\partial \left( v \left( pe_t + p + \frac{1}{(\gamma - 1)M_\infty^2} \right) \right)}{\partial y} = 0
\]

where asterisks are dropped for notational convenience. The corresponding non-dimensional free-stream conditions now become

\[
\rho_\infty = 1 \quad u_\infty = 1 \quad p_\infty = 0 \quad (pe_t)_\infty = 0.5
\]

It is important to note that in the present formulation, the free-stream conditions and the conservation of momentum equation do not contain any singularity. However, the cancellation problem still exists in the conservation of energy equation for zero Mach number flows. It is next removed by introducing a new preconditioning matrix. The eigenvalues of the present set of equations are

\[
\lambda_1 = u - c \quad \lambda_2 = u \quad \lambda_3 = u \quad \lambda_4 = u + c
\]
where the modified speed of sound, $c$, is given by

$$c = \sqrt{\frac{\gamma p}{\rho} + \frac{1}{\rho M_\infty^2}}$$

It should be noted that as $M_\infty$ goes to zero, the $u + c$ and $u - c$ eigenvalues still go to ±∞. The present formulation is developed by first relaxing the time derivative terms in both the conservation of mass and the energy equations. It is achieved simply by dividing them by $M_\infty^2$ or multiplying the remaining terms by $M_\infty^2$.

$$\begin{align*}
\frac{\partial \rho}{\partial t} + M_\infty^2 \frac{\partial (\rho u)}{\partial x} + M_\infty \frac{\partial (\rho v)}{\partial y} &= 0 \\
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} + \frac{\partial (\rho uv)}{\partial y} &= 0 \\
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho uv)}{\partial x} + \frac{\partial (\rho v^2 + p)}{\partial y} &= 0 \\
\frac{\partial (\rho e_t)}{\partial t} + M_\infty^2 \frac{\partial \left( u \left( \rho e_t + p + \frac{1}{(\gamma - 1) M_\infty^2} \right) \right)}{\partial x} + M_\infty^2 \frac{\partial \left( v \left( \rho e_t + p + \frac{1}{(\gamma - 1) M_\infty^2} \right) \right)}{\partial y} &= 0
\end{align*}$$

(7)

At the limiting case of zero Mach number, the equation set becomes

$$\begin{align*}
\frac{\partial p}{\partial t} &= 0 \\
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} + \frac{\partial (\rho uv)}{\partial y} &= 0 \\
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho uv)}{\partial x} + \frac{\partial (\rho v^2 + p)}{\partial y} &= 0 \\
\frac{\partial (\rho e_t)}{\partial t} + \frac{1}{(\gamma - 1)} \frac{\partial u}{\partial x} + \frac{1}{(\gamma - 1)} \frac{\partial v}{\partial y} &= 0
\end{align*}$$

(8)

The relaxed, preconditioned conservation of mass equation diminishes for incompressible flows. Yet, the conservation of energy equation becomes bounded and provides the divergence free velocity constraint. However, it can be shown that the eigenvalues of the equation set given previously are rather complex in comparison to the ACM formulation. It is observed that the preconditioned conservation of energy equation given previously and the modified conservation of mass equation in the ACM method (Equation (5)) are similar with the exception that the time rate of change of total energy is considered in the former instead of that of artificial density or pressure in the latter. In the present formulation, the ideal gas equation, Equation (2), relates the pressure to the total energy. Its differentiation with respect to time for incompressible flows provides

$$\frac{1}{(\gamma - 1)} \frac{\partial p}{\partial t} = \left( \frac{\partial (\rho e_t)}{\partial t} - u \frac{\partial (\rho u)}{\partial t} - v \frac{\partial (\rho v)}{\partial t} \right)$$

(9)
When Equation (9) is substituted into Equation (8), the modified conservation of mass equation of the original ACM formulation can be recovered by subtracting the $\frac{\partial (\rho u)}{\partial t}$ and $\frac{\partial (\rho v)}{\partial t}$ terms:

$$
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} + \frac{\partial (\rho u v)}{\partial y} = 0 \\
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho u v)}{\partial x} + \frac{\partial (\rho v^2 + p)}{\partial y} = 0 \\
\frac{\partial (\rho e_t)}{\partial t} - u \frac{\partial (\rho u)}{\partial t} - v \frac{\partial (\rho v)}{\partial t} + \frac{1}{(\gamma - 1)} \frac{\partial p}{\partial t} = 0 \\
\frac{\partial (u \left( M^2_{\infty} \rho e_t + M^2_{\infty} p + \frac{1}{(\gamma - 1)} \right) )}{\partial x} + \frac{\partial \left( v \left( M^2_{\infty} \rho e_t + M^2_{\infty} p + \frac{1}{(\gamma - 1)} \right) \right)}{\partial y} = 0
$$

Although the addition/subtraction of such unsteady terms affects the transient numerical solution, once the solution converges to a steady state, the divergence free velocity field is satisfied, which is the general character of ACM and other preconditioning methods for steady flows. The final form of the preconditioned system of Euler equations now becomes

$$
\frac{\partial \rho}{\partial t} + M^2_{\infty} \frac{\partial (\rho u)}{\partial x} + M^2_{\infty} \frac{\partial (\rho v)}{\partial y} = 0 \\
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} + \frac{\partial (\rho u v)}{\partial y} = 0 \\
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho u v)}{\partial x} + \frac{\partial (\rho v^2 + p)}{\partial y} = 0 \\
\frac{\partial (\rho e_t)}{\partial t} - (1 - M^2_{\infty}) u \frac{\partial (\rho u)}{\partial t} - (1 - M^2_{\infty}) v \frac{\partial (\rho v)}{\partial t} + \\
\frac{\partial \left( u \left( M^2_{\infty} \rho e_t + M^2_{\infty} p + \frac{1}{(\gamma - 1)} \right) \right) }{\partial x} + \frac{\partial \left( v \left( M^2_{\infty} \rho e_t + M^2_{\infty} p + \frac{1}{(\gamma - 1)} \right) \right) }{\partial y} = 0
$$

The equation set given previously can also be expressed in terms of a preconditioning matrix for the Euler equations:

$$
\frac{\partial Q}{\partial t} + \Gamma \left( \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} \right) = 0
$$

where

$$
Q = \begin{pmatrix}
\rho \\
\rho u \\
\rho v \\
\rho e
\end{pmatrix}
E = \begin{pmatrix}
\rho u \\
\rho u^2 + p \\
\rho u v \\
u (\rho e + p + \frac{1}{(\gamma - 1) M^2_{\infty}})
\end{pmatrix}
F = \begin{pmatrix}
\rho v \\
\rho v u \\
\rho v^2 + p \\
v (\rho e + p + \frac{1}{(\gamma - 1) M^2_{\infty}})
\end{pmatrix}
$$

and $\Gamma$ is the Mach-uniform preconditioning matrix:

$$
\Gamma = \begin{bmatrix}
M^2_{\infty} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & (1 - M^2_{\infty}) u & (1 - M^2_{\infty}) v & M^2_{\infty}
\end{bmatrix}
$$
The flux Jacobian matrix, $J$, and the eigenvalues of the preconditioned system in the $n-t$ coordinate system are obtained as

$$J = \begin{pmatrix} 0 & M_{\infty}^2 & 0 \\ \frac{(\gamma-1)u - u_n^2}{-u_n u_t} & (3-\gamma)u_n & -(\gamma - 1)u_t \\ \frac{u_n (A + (\gamma - 1) u_n^2)}{(\gamma - 1)} & B - \frac{(M_{\infty}^2 - 2)q}{2} - H u_n u_t & (H + 1)u_n \end{pmatrix}$$

where $\lambda_1 = u_n$, $\lambda_2 = u_n M_{\infty}^2$, $\lambda_3 = u_n - c$, $\lambda_4 = u_n + c$

The proposed preconditioning method recovers ACM at the limiting case of incompressible flows, which differentiates it from the other well-known preconditioning methods [17, 18]. It should be noted that the modified speed of sound is bounded for $-1 < M_{\infty} < 1$ and at $M_{\infty} = 0$ has a value of $\sqrt{\frac{\gamma p M_{\infty}^2 + 1}{\rho} + (1 - M_{\infty}^2) u_n^2}$. The condition number becomes $(u + c)/|u - c| = 5.8$, which is the same as that of ACM with $\delta = 1$ and approximately twice the value of the Choi–Merkle preconditioner. In ACM, the condition number can be reduced to approximately 2.6 when $\delta$ is defined as proportional to flow velocity [19]. A similar approach can be considered in the present formulation in order to improve the convergence rate. On the other hand, the present preconditioning should not be employed for supersonic flows for which the modified speed of sound may have a complex value with an imaginary component.

### 2.1. Flow solver

The Mach-uniform preconditioner developed is implemented in an in-house flow solver. Because the main objective of this study is to present a Mach-uniform preconditioning method, the flow solver employs point Gauss–Seidel algorithm for temporal discretization because of its simple implementation. The preconditioning method developed can be employed in fully implicit and line implicit solution algorithms with multi-grid acceleration as well. Second-order Roe’s approximate Riemann solver for convective flux evaluations is implemented for spatial discretization. The slope limiter of Barth and Jespersen [20] is employed in the second-order reconstruction of flow variables. The Roe fluxes are evaluated with the general formula:

$$\frac{1}{2} (F_L + F_R - T|\lambda|T^{-1}DQ)$$

where $F_L$ and $F_R$ are left and right state fluxes, respectively, $\lambda$ and $T$ are eigenvalues and the eigenvectors of the preconditioned jacobian matrix formed by the Roe averaged flow variables, respectively. The eigenvectors and its inverse are obtained by using the commercial symbolic algebra software MAPLE (Maplesoft, Waterloo, Ontario, Canada) and given later in the $n-t$ coordinate system.
system. \(u_n\) and \(u_t\) again represent normal and tangential velocity components along the \(n\) and \(t\) coordinates, respectively.

\[
\lambda = \begin{pmatrix}
0 & 1 & \lambda_1 & \lambda_2 \\
0 & u_n & \lambda_3 & \lambda_4 \\
1 & 0 & \lambda_5 & \lambda_6 \\
\frac{1}{2}u_t & -\frac{1}{2}u_t^2 & \lambda_7 & \lambda_8
\end{pmatrix}
\]

\[
T = \begin{pmatrix}
0 & 1 & \frac{M_{\infty}^2}{u_n - c} & \frac{M_{\infty}^2}{u_n + c} \\
0 & u_n & \frac{u_n}{u_n - c} & \frac{u_n}{u_n + c} \\
1 & 0 & \frac{u_n (M_{\infty}^2 - 1)}{c} & \frac{M_{\infty}^2}{c} \\
\frac{1}{2}u_t & -\frac{1}{2}u_t^2 & \frac{2u_t (M_{\infty}^2 - 1)}{c} & \frac{2u_t}{c}
\end{pmatrix}
\]

\[
T^{-1} = \begin{pmatrix}
\frac{u_t (y-1)q}{2c^2} & \frac{u_t u_n H}{c^2} & \frac{(y-1)u_n^2 + c^2}{c} & \frac{(y-1)u_t}{c} \\
\frac{M_{\infty}^2 (y-1)q - 2(M_{\infty}^2 - 1)u_n^2}{2FG} & \frac{u_n H - c}{2F} & \frac{M_{\infty}^2 (y-1)u_t}{FG} & \frac{M_{\infty}^2 (y-1)}{FG} \\
\frac{(y-1)q + 2u_n c}{4Fc} & \frac{u_n H + c}{2F} & \frac{M_{\infty}^2 (y-1)u_t}{2Fc} & \frac{M_{\infty}^2 (y-1)}{2Fc} \\
\frac{(y-1)q - 2u_n c}{4Fc} & \frac{u_n H - c}{2F} & \frac{(y-1)u_t}{2Fc} & \frac{(y-1)u_t^2}{2Fc}
\end{pmatrix}
\]

where \(c\), \(q\), and \(H\) are defined in Equation (12) and

\[
D = \frac{(M_{\infty}^2 - 1) u_n^2 + c^2}{(y-1)} - \frac{(M_{\infty}^2 - 1) c^2}{2(y-1)}
\]

\[
E = u_n c - \frac{(M_{\infty}^2 - 1) u_n u_t^2}{c}
\]

\[
F = (M_{\infty}^2 - 1) u_n + c
\]

\[
G = (M_{\infty}^2 - 1) u_n - c
\]

### 2.2. Boundary conditions

Boundary conditions play a significant role both on the convergence characteristics and on the accuracy of flow solvers. In this study, for subsonic external flows, the following standard boundary conditions, which are also employed in ACM, are implemented:

\[
\begin{align*}
\rho &= \rho_\infty \\
\rho u &= (\rho u)_\infty \\
\rho v &= (\rho v)_\infty \\
P &= P_{\text{ext}} \quad \text{for inflow}
\end{align*}
\]

\[
\begin{align*}
\rho &= \rho_\text{ext} \\
\rho u &= (\rho u)_\text{ext} \\
\rho v &= (\rho v)_\text{ext} \\
P &= P_\infty \quad \text{for outflow}
\end{align*}
\]

where \(\infty\) and \(\text{ext}\) correspond to free-stream and extrapolated values from the solution domain, respectively. It should be noted that for high subsonic flows, these boundary conditions result in significant variations on the inflow total energy, especially when high level of disturbances interacts with near-field boundaries. For such cases, including channel flow, the entropy and the stagnation enthalpy at the inflow and the pressure at the outflow are taken as the free-stream values. The remaining variables are extrapolated from the interior flow solution. For supersonic flows, all of the flow variables are taken as the free-stream values at the inflow and extrapolated from the interior solution at the outflow.

### 3. RESULTS AND DISCUSSION

The present Mach-uniform preconditioning method developed is validated for the inviscid flows in a channel with a circular bump and for the external flows over a NACA0012 airfoil for a wide range of Mach numbers. Except the supersonic flow cases, the preconditioner is activated in all the flow solutions. For comparison, compressible flow solutions without the preconditioner are also obtained.
Figure 1. Computational grids employed for channel flows with circular bumps.

Figure 2. Pressure distributions for channel flows with circular bumps.

Figure 3. Mach number distributions on upper and lower channel walls for different Mach numbers.
3.1. Flows in a channel with a circular bump

Flows in a two-dimensional channel with a circular bump are studied as the first validation case. Three different inflow Mach numbers, 0.01, 0.675 and 1.40, are considered. The circular bump height is taken as 10% for the subsonic and transonic flow cases and 4% for the supersonic flow case in accordance with the reference study performed by Ni [21]. The computational grid is of 252 × 54 size in both cases as shown in Figure 1. The pressure distributions in the channel and the Mach number variations on the top and the bottom channel walls are given in Figures 2 and 3. As seen in the figures, the shocks and the shock reflections in the transonic and the supersonic flow cases are resolved sharply, and the present predictions compare well against the reference study [21], which is based on a multi-grid flow solver. The convergence histories of the present method and the compressible flow solution are given in Figure 4. The compressible solution does not converge for the low Mach number case, whereas the present method developed achieves a linear convergence rate. On the other hand, for the transonic case, the convergence rate of the present solution initially behaves similar to that of the compressible solver but later deteriorates slightly. Such a deterioration is attributed to the far-field boundary conditions, which are currently not non-reflective. For non-reflective boundary conditions, the characteristic form of the preconditioned Euler equations should first be derived and employed properly.

3.2. Flows over a NACA0012 airfoil

As the second validation case, inviscid flows over a NACA0012 airfoil profile are studied extensively. A grid dependency test is first conducted for a wide range of Mach numbers 0.0 ≤ M∞ ≤ 0.7 using three C-type grids for which the grid resolution is quadrupled progressively. The results presented in Table I show the drag coefficient reduction for all Mach numbers as the grid resolution increases where the correct value is zero for the inviscid solutions.

The convergence rate of the present preconditioner is next compared against two other similar studies; one of which is based on Rossow’s formulation [17], and the other one is based on Turkel’s formulation [22]. In these comparison cases, the inviscid flows over NACA0012 airfoil are computed using Roe upwinding and Runge–Kutta time stepping with similar mesh sizes. The reference [17] employs first-order spatial discretization and 10 sub-iterations in each Runge–Kutta stage. As observed in Figure 5(a), the present solution, which now employs first-order spatial discretization

Table I. Grid dependency study for drag convergence.

<table>
<thead>
<tr>
<th>Mach number</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grid</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>180 × 30</td>
<td>0.00148</td>
<td>0.00148</td>
<td>0.00148</td>
<td>0.00149</td>
<td>0.00149</td>
<td>0.00149</td>
<td>0.00149</td>
<td>0.00143</td>
</tr>
<tr>
<td>360 × 60</td>
<td>0.00091</td>
<td>0.00091</td>
<td>0.00091</td>
<td>0.00091</td>
<td>0.00090</td>
<td>0.00089</td>
<td>0.00087</td>
<td>0.00084</td>
</tr>
<tr>
<td>720 × 120</td>
<td>0.00080</td>
<td>0.00080</td>
<td>0.00080</td>
<td>0.00080</td>
<td>0.00079</td>
<td>0.00078</td>
<td>0.00076</td>
<td>0.00073</td>
</tr>
</tbody>
</table>
for comparison, has initially a better convergence rate than that of reference [17] in spite of the sub-iterations employed in the latter. The reduced convergence rate following the first two orders of reduction in the present study may be attributed to the far-field boundary conditions employed, which are not non-reflective as stated earlier. In the next comparison with reference [22], in which both solutions employ second-order spatial discretization, the present method has a significantly better convergence rate than reference [22] as shown in Figure 5(b).

The incompressible and a low Mach number of $M_1 = 0.001$ flow solutions obtained on the $360 \times 60$ grid are compared to the incompressible flow solutions obtained by the ACM formulation and by a panel method in Figure 6. As shown, the present method predicts the surface pressure distribution accurately. The residual histories given in Figure 6 show that the convergence rate of
the present preconditioner for $M_\infty = 0.01$ and $M_\infty = 0$. flows is almost identical and about the same as the ACM method.

The preconditioned solution at $M_\infty = 0.01$ is compared against the non-preconditioned solution in Figure 7. Although the pressure distributions, in general, agree well, the non-preconditioned flow solution exhibits the expected slow convergence and regional stability issues. The stability of the non-preconditioned solution deteriorates especially at the stagnation region because of numerical pressure oscillations.

The accuracy and performance of the present preconditioner are also evaluated at high subsonic and transonic flow conditions. The flow solutions with and without the preconditioner are first presented for $M_\infty = 0.50$. The Mach number distributions and the residual histories are compared in Figure 8. As shown, the accuracy and the convergence acceleration of the preconditioned solution at a high subsonic Mach number solution do not deteriorate for external flows. Next, the transonic flow at $M_\infty = 0.85$, which is a widely used AGARD [23] test case for compressible flow solvers, is considered. As shown in Figure 9, the predictions of the shock locations with and without the preconditioner are about the same, and both agree well with the AGARD data. In addition, the convergence rate of the preconditioned solution still does not deteriorate at transonic flow conditions. The convergence characteristics of the present preconditioner are presented all together for a range of subsonic Mach numbers in Figure 10. It is observed that almost a uniform convergence rate is achieved for all the subsonic Mach number flows up to $M_\infty = 0.5$ including $M_\infty = 0$ case, which shows the Mach-uniform solution efficiency of the present preconditioner.

![Figure 8](image8.png)

Figure 8. (a) Pressure distributions and (b) residual histories for flow over NACA0012 airfoil at $M = 0.5$ and $\alpha = 1^\circ$.

![Figure 9](image9.png)

Figure 9. (a) Pressure distributions and (b) residual histories for flow over NACA0012 airfoil at $M = 0.85$ and $\alpha = 1^\circ$. 
4. CONCLUSIONS

In this study, a novel Mach-uniform preconditioning method is developed for the solution of Euler equations. The preconditioned equations provide stability and convergence acceleration for the solution of low subsonic flows including incompressible flows at $M_{\infty} = 0$. The Mach-uniform preconditioner is validated for channel flows with a circular bump and external flows around an airfoil for a wide range of Mach number flows including incompressible flows. It is shown that the preconditioner prevents the instability of compressible flow solvers at very low Mach numbers including the zero Mach number case and provides a uniform convergence rate for a wide range of subsonic flows where $0 \leq M_{\infty} < 0.5$. The present preconditioned system of equations is equivalent to the ACM formulation at the limiting case of $M_{\infty} = 0$. Yet, the Mach-uniform convergence rate achieved in the present formulation is similar to that of ACM not only for incompressible flows but also for low subsonic flows as well.

REFERENCES


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